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18.440: Probability and Random Variables
Problem Set 3

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Correctness Statement

While there is always a chance of errors in any body of the work, everything I achieve here will be self taught—which means the risk of errors is perhaps higher than usual. If, for some reason, one finds themself on my page tempted to use my work, please first check that it is indeed correct. If you find an error, email me and I'll fix it :)

Note: The majority of the below problems are from *A First Course in Probability* 8th ed. by Sheldon Ross.

1. **Problem 26.** Suppose that 5 percent of men and .25 percent of women are color blind. A color-blind person is chosen at random. What is the probability of this person being male? Assume that there are an equal number of males and females. What if the population consisted of twice as many males as females?

   Solution. Suppose there are x men and x women. Then there are $0.05x$ men that are color blind and $0.0025x$ women that are color blind. Hence, there are $0.0525x$ people that are color blind. Therefore, the probability that the randomly selected color blind person is male is

$$\frac{0.05x}{0.0525x} = \frac{500}{525} = \frac{20}{21} \approx 0.95.$$

   If there are twice as many males as females, the probability will only increase. Now there are $0.05 \cdot 2x$ men that are color blind and there are still $0.0025x$ women that are color blind. So the probability is now

$$\frac{0.1x}{0.1025x} = \frac{1000}{1025} = \frac{40}{41} \approx 0.98.$$

   Edit: When I started this problem set, I was still very...relative frequency happy. By which I mean, the lense through which I solved problems always defaulted to counting desired outcomes and total outcomes. In this case, I actually think that's still a valid (and much simpler) method. However, I think we could achieve the same results using **Bayes's rule**. Let C be the event that a person is color blind, M be the event that a person is a man (notice M and M^c partition Ω). We could then use Bayes's rule to find $P(M|C)$.

2. **Problem 43.** There are 3 coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up heads 75 percent of the time. When one of the 3 coins is selected at random and flipped, it shows heads. What is the probability that it was the two-headed coin?

   Solution. Consider the following events:

   - W : the coin selected is two-headed (I selected W because T usually denotes tails, and w is the second letter in two)
   - F : the coin selected is fair
   - B : the coin selected is biased
   - H : the coin flip lands on heads (we don't need an event for tails)

   We wish to find $P(W|H)$. Observe that

$$\begin{aligned} P(W|H) &= \frac{P(W \cap H)}{P(H)} \\ &= \frac{P(W \cap H)}{P(W \cap H) + P(F \cap H) + P(B \cap H)} \\ &= \frac{1}{1 + 0.5 + 0.75} \\ &= \frac{1}{2.25} \\ &= \frac{100}{225} = \frac{4}{9} \approx 0.44. \end{aligned}$$

If the above isn't clear, the denominator $P(H)$ is rewritten using the total probability law, and the probabilities themselves are implicitly given in the problem statement. Thus, $P(W|H) = 4/9$ and we are done.

3. **Problem 47.** An urn contains 5 white and 10 black balls. A fair die is rolled and that number of balls is randomly chosen from the urn. What is the probability that all of the balls selected are white? What is the conditional probability that the die landed on 3 if all the balls selected are white?

   Solution. Let R_i and A be events. The event R_i corresponds to the die rolling the value i for $1 \leq i \leq 6$ and the event A corresponds to all drawn balls being white. We can find $P(A)$ using the total probability law:

$$P(A) = \sum_{i=1}^{6} P(R_i)P(A|R_i).$$

   Next, notice that $P(A)$ and $P(R_i)$ are independent so $P(A|R_i)$ is simply

$$\frac{\binom{5}{i}}{\binom{15}{i}}.$$

   We now have

$$\begin{aligned} \sum_{i=1}^{6} P(R_i)P(A|R_i) &= \frac{1}{6}\left(\frac{5}{15} + \frac{10}{105} + \frac{10}{455} + \frac{5}{1365} + \frac{1}{3003}\right) \\ &= \frac{5}{66} \approx 0.076 \end{aligned}$$

   so $P(A) = 5/66$. For the second part of the problem we need to find $P(R_3|A)$ which can be achieved with

$$\begin{aligned} P(R_3|A) &= \frac{P(R_3 \cap A)}{P(A)} \\ &= \frac{P(R_3)P(A|R_3)}{P(A)} \\ &= \frac{\frac{1}{6} \cdot \frac{10}{455}}{\frac{5}{66}} \\ &= \frac{22}{455} \approx 0.048. \end{aligned}$$

4. **Problem 76.** Suppose that E and F are mutually exclusive events of an experiment. Show that if independent trials of this experiment are performed, then E will occur before F with probability

$$\frac{P(E)}{P(E) + P(F)}.$$

   Solution. Let n denote the number of repetitions that occur prior to the event E occuring. If $n = 1$, then E occurs immediately. In all other cases, prior to the nth repetition, E does not occur and F does not occur, which means something in $E^c \cap F^c$ occurs. Let $G = E^c \cap F^c$ and let G_i denote the ith occurence in a row of G. Then, the probability that E occurs before F in n repetitions is

$$P(G_1 \cap G_2 \cap \cdots \cap G_{n-1} \cap E) = P(G_1)P(G_2|G_1)P(G_3|G_2 \cap G_1) \cdots P(E|G_1 \cap \cdots \cap G_{n-1}).$$

I think[1] because E and F are exclusive, E^c and F^c should also be exclusive. Thus, $G = P(E^c)P(F^c)$. I think we can then rewrite the probability that E occurs before F in n repetitions as

$$P(G_1 \cap G_2 \cap \cdots \cap G_{n-1} \cap E) = P(G)^{n-1}P(E).$$

We're making good progress, but we still aren't done. We don't really care if E occurs before F in a single repetition, two repetitions, or arbitrarily many repetitions. All cases are desirable, so the total probability that E occurs before F is the union of desired probabilities over all repetitions. These probabilities are disjoint, so we may simply sum them up. Let p be the total probability that E occurs before F, and then we have

$$\begin{aligned} p &= \sum_{i=0}^{\infty} P(G)^i P(E) \\ &= P(E)\frac{1}{1-P(G)} && \text{[convergent geo series]} \\ &= \frac{P(E)}{P(G^c)} \\ &= \frac{P(E)}{P(E \cup F)} \\ &= \frac{P(E)}{P(E)+P(F)}. \end{aligned}$$

5. **Theoretical Exercise 1.** Show that if $P(A) > 0$, then

$$P(AB|A) \geq P(AB|A \cup B).$$

Solution. From the definition of conditional probability we have

$$P(A \cap B|A) = \frac{P(A \cap (A \cap B))}{P(A)} = \frac{P(A \cap B)}{P(A)}$$

and

$$P(A \cap B|A \cup B) = \frac{P((A \cap B) \cap (A \cup B))}{P(A \cup B)} = \frac{P(A \cap B)}{P(A \cup B)}.$$

[1] If I were submitting this for a grade, I would cite a theorem or provide a proof, but I'm not submitting for a grade and I also want to move on to random variables. Therefore, we will sacrifice rigor to save time.

Notice that the numerators are the same, but the denominators are not. Moreover, $A \leq A \cup B$ so we must have

$$\frac{P(A \cap B)}{P(A)} \geq \frac{P(A \cap B)}{P(A \cup B)} \quad \Rightarrow \quad P(A \cap B|A) \geq P(A \cap B|A \cup B)$$

as desired.

6. OMITTED
7. OMITTED
8. OMITTED

   If I have time, I'll come back to 6-8 someday. For now, I need to focus on my REU applications and I'd also like to move on to random variables.