Concept

I want to cover the entire AOPS curriculum in 11 months (roughly 320 days). The curriculum is about 4800 pages so if I average 15 pages a day I can do it.

This will be a lot of work so I want to record my progress.

As it turns out, this document became less of a "record my progress" document and more of a "problem-solving notes" document. At some point I need to clean it up, but for now, it should be noted that there are borrowed examples and blocks of code from AoPS throughout this entire document. There are sections where I will make explicit note of such code or examples, but overall, even my own original thoughts would not have been possible without first working through AoPS material. As such, those impressed with anything other than my discipline/committment, should give credit to the people that created (or collected[1]) the intellectual property: the amazing team at AoPS.

Contents

[1] many AoPS are repurposed problems from various math competitions and are always cited as such

Book Notes

1 Prealgebra

1.1 Properties of Arithmetic

We all learn the basics of Arthimetic but few are lucky enough to know how much we can really do with it. In the first chapter we learn how to get the most out of arthimetic. I believe this is best demonstrated with Gauss's sum of pairs but there are other clever examples. For instance, we can separate numbers into other numbers to make computation easier. Computing this multiplication

$$9,999 \times 23$$

would be a pain if we don't recognize that we can simply rewrite it like so:

$$9,999 \times 23 = (10,000 - 1) \times 23 = 230,000 - 23 = 229,977.$$

Problems to revisit:

i. Clever addition makes for easy multiplication.

   I saw a solution to this problem that somehow works, but I can't really explain how.

   The book simply rewrote 999...999 as $(10000...0 - 1)$ and then computed the multiplication, got the number, and added the digits. We got the same answer, but in very different ways. Except, I think they're not different?

   Ex: $99 \cdot 44 = 11^2 * 36 = (36 + 360) \cdot 11 = 3600 + 360 + 360 + 36 = 3600 + 720 + 36 = 4356 \Rightarrow \sum digits = 9 \cdot 2$

   notice though, that 99 times 44 is not 3636 but the sum of digits is the same. CLEAN UP YA NOTES BOOOOIIII

ii. Clever use of ch1 in ch 3

   Edit: 7/7/23 no longer needed for review.

iii. ch1 + ch4 = good problem solving

1.2 Exponents

Problems to revisit:

i. Consecutive squares are easy!

   Edit: 7/7/23 long sense not needed.

ii. Sometimes having pen and paper is nice.

iii. Digit counting isn't so hard.

iv. No hint for you

This is worth revisiting a few times. There is an easy way of approaching the problem, and a not so easy way. Good luck.

Edit: It's still worth revisiting lmfao.

v. Removing hints is probably a good idea lmfao

Needs to be in rotation every time until it's obivous

vi. No hint!

vii. Algebra go brrr

I think a harder problem that's more valuable is:

Factor $a^3 - b^3$ into a product of two expressions.

Alternatively:

Factor $a^3 + b^3$ into a product of two expressions.

Edit: I think what I was orignally getting at is the idea of differences and sums of powers. Generalizing that (which we do in intermediate algebra) is better.

1.3 Number Theory

i. Work smarter not harder

ii. booga booga

iii. Interesting topic: **Sieve of Eratosthenes** (fast algorithm for generating primes)

iv. Great example of how to use LCM

Even better: hibbity bippity

v. Remainders are your friend

vi. lol get trolled

Very clever problem

vii. Good problem

Yeaaaah this one needs to stay in rotation

viii. read carefully. I'll say that again. READ CAREFULLY

ix. This one needs to be attempted until it's second nature

x. don't forget about relatively prime numbers!

This needs to be in rotation

Need to make some notes about LCM vs. GCD

LCM (find prime factorization, then largest powers of all primes needs to be present)

LCM biggest powers GCD smalles powers

To test for divisibility by a composite number, we can test for divisibility by two relatively prime numbers whose product is the composite number. See here:

https://artofproblemsolving.com/ebooks/prealgebra-ebook/par/96342

1.4 Fractions

1. a problem
2. another problem
3. a third problem lmao

1.5 Linear Equations and Inequalities

If $x > y$ and $a > b$, then $x + a > y + b$. If $x \geq y$ and $a \geq b$, then $x + a \geq y + b$. In other words, if we have two inequalities, then the sum of the larger sides of the inequalities is greater than the sum of the smaller sides of the inequalities.

Useful for explaining the reverse rule in inequalities: https://artofproblemsolving.com/ebooks/prealgebra-ebook/par/97803

(don't forget, dividing by negative number is multiplying by a negative number)

if you do it correctly, it's trivial

1.6 Decimals

We can convert repeating decimals by using a trick. Let some repeating decimal d be

$$d = 0.321321321321321321...$$

Notice that $1000d - d = 321$ so we have

$$999d = 321 \Rightarrow d = 321/999.$$

Be careful about decimals in which not every number is repeating:

$$x = 0.28888 \Rightarrow 10x - x = 2.88888... - 0.28888888.... = 2.6 \Rightarrow x = 26/90 = 13/45.$$

VERY clever

1.7 Ratios, Conversions, Rates

Need to do a little more practice with ratios. Remember, ratios are relative. Therefore, we neecd to account for total by "adding both (or all) parts of ratio together".

sneaky

more involved than it may seem

throwback to middle/elementary school

get trolled again lol

1.8 Percents

Do this with variables only

1.9 Square Roots

Something to revisit at a later date: Geometric Mean

Ask Jeremy/CJ about this

don't approximate; get an exact answer

do with variables

be more dilligent

use inequalities

you should be able to prove this

1.10 Angles

DON'T use triangles

disappointed you didn't solve this first try

Solve without hint!

LOL don't just guess. You will be very wrong

1.11 Perimeter and Area

Surprisngly simple, yet powerful

Solve wihthout using Pythagorean Theorem

you should have been able to do this faster

If Orthocenters don't appear in the Geometry text, you need to go back and do this

I don't understand this at all

Didn't fully attempt due to fatigue

LOL ITS NOT INFINITe

space can spin, bro

bruh moment

gosh darn it

This must rely on the conclusions we made from the paint exercise...

1.12 Right Triangles and Quadrilaterals

Interesting proof of Pythagorean Theorem:

proof go brrr

Probably a helpful video:

Classifying Quadrilaterals

Problems to revist:

i. clever tool
ii. it helps if you don't make ridiculous errors, dummy
iii. Think more, do less

iv. Don't forget to read whole problem

v. lol okay

vi. you're losin it, man

vii. time to take a break

Not sure the last three need to be on the list, but whatever

viii. Try to prove this

ix. CHALLENGE AND REVIEW PROBLEMS

1.13 Data and Stats

N/A

1.14 Counting

Problems to revisit:

i. "don't do shortcuts unless you can explain why they work" lmao

ii. thinking was just wrong :P

iii. you don't know how to count

iv. READ, DUDE. CMON!

v. booga booga

vi. zero isn't positive

vii. reeeeeadddd

viii. casework go brrr

ix. close, but not close enough

x. fantastic casework problem

xi. My method is correct, their method is more clever

xii. wow

xiii. successful vs. possible

xiv. cards go fwhip fwhip fwhip

xv. I counted...literally. How would you generalize?

xvi. counting is hard

xvii. I got right answer, but work was messy

End of Pre-Algebra!

2 Intro to Algebra

2.1 Follow the Rules

exponents

more exponents

radicals

2.2 x Marks the spot

fancy expression work

2.3 One Variable Linear Equations

an idea

When will this be helpful, is my question. I'm sure there's probably an example, but I'd like to see it.

Be careful with even powers

a trick for consecutive numbers

Work smart, not hard

Look for the clever solution without using the hint

DOn't overcomplicate

2.4 More Variables

you need to get better at making sure there isn't just one solution

2.5 Multi-Variable Linear Equations

when it doubt, write it out

lol be more careful

"analytical symmetry" go brrrr

don't need to solve to solve

Should probably revist 5.6

READ CAREFULLY

Intro to Alg? More like Advanced Linear Alg lmfaooo

hardest problem to date (Jan 29)

2.6 Ratios and Percents

Problems to revisit:

i. ur trash at ratios my guy

ii. clock

iii. Why am I so bad at ratios????

iv. attempt entire problem without help

v. what da heck

vi. make sure you get part b right

vii. I fell for bogus solution lol

viii. if your life depended on your ability to read, you would die

ix. Try to solve with ratios instead of algebra

x. haha stoopid

xi. you're improving, nice

xii. instincts were correct, but you didn't finish the problem!
very nice problem

You MUST revist 6.2 (challenging ratio problems)

2.7 Proportion

How would this problem be solved using rates in Pre-Algebra?

Do when not tired lmao

swag money

oof

Do Challenge Problems

2.8 Graphing Lines

Talk to Jeremy lol

Good methods:

i. remember ratios

ii. one line is better than two

iii. use slope instead of algebra

Problems to do again:

i. hehe algebra go brrr

ii. should be able to quickly grasp concept

iii. REMINDS ME OF PUTNAM TRICKS LETDS GOOOOOO

Not one of the most beautiful problems I've seen so far, but still one of my favorites. (2/3/23)

iv. Not easy to spot the trick

2.9 Inequalities

Inequalities can be used to convey values of a variable when it isn't practical to substitute in all the values you wish to describe. As such, inequalities can (sometimes) be useful in problems where you need to prove that one solution is the only solution.

reciprocals

Problems to do again:

i. lol did wrong thing and got right answer...but luck may not be so kind next time

ii. use smallest powers possible

iii. You really should be able to do this

iv. stop doing stuff in your head!

v. Be careful

vi. you're tired. Go to sleep

2.10 Quadratic Equations pt. 1

Clever/Important Ideas:

i. If you can't explain why this is true, you need to revisit the material

ii. Same here. You should be able to explain this

iii. Rough summary of ideas above, but it's forgetting something quite powerful

iv. Don't forget to check multiple cases. For instance, is zero a trivial solution?

v. Don't forget to check if solution is an actual solution (ex: if solution makes a denominator zero, it obviously isn't a solution).

vi. Anything loosely resembling a quadratic is factorable if you're crazy enough

vii. Using systems of equations, we can separate a problem into easier parts.

$$
\begin{aligned}
&43 - 30\sqrt{2} = a^2 + 2b^2 + 2ab\sqrt{2} \\
&\Rightarrow \text{system of equations:} \\
&\quad a^2 + 2b^2 = 43 \\
&\qquad\quad 2ab = -30
\end{aligned}
$$

Problems to do again:

i. duuuuuuude. You CANNOT mess that up. CMON!

ii. Be careful. You don't want to accidentally divide by zero

iii. Challenge: Factor without guessing numbers. Use logic to elliminate possibilities before trying them

iv. SLOOOOOWWWWW down. You got the right answer, but you worked way harder than necessary.

v. CASES!

vi. Given the section title, what method should you use to solve this problem? Solve the problem.

vii. Get a number

viii. Your method was correct, but you didn't need to work that hard to use that method.

ix. sort of right answer, wrong method. Follow their steps

x. Iiiiii just wanna play it riiiiiiiiight

xi. Weeeeeeeee are gonna get there toniiiiiiight *wob wob*

xii. If at first you don't succeed...

xiii. I could answer this[2] on my own but I'm guessing they will later. If not, we should prove my hypothesis (that it chains in a manner similar to induction).

$$
\begin{aligned}
&(x-p)(x-q)(x-r)\\
&=[(x-p)(x)+(x-p)(-q)](x-r)\\
&=(x^2-px-qx+pq)(x-r)\\
&=(x^2-px-qx+pq)(x)+(x^2-px-qx+pq)(-r)\\
&=x^3-px^2-qx^2+pqx-rx^2+rpx+qrx-pqr\\
&=x^3-(p+q+r)x^2+(pq+qr+rp)x-pqr.
\end{aligned}
$$

xiv. If a problem does not state "in terms of...", the solution is probably a number/set of numbers.

xv. Keeping numbers in your equation might help, silly

xvi. If you guess, you must prove that you are not missing any!

xvii. infinite roots aren't so scary if you're clever

Questions:

1. How in teh fricken heck am I supposed to recognize that 45*84=3780??? Also, how many ways can the prime factors be combined to make the factors of 3780?

   Edit: 7/10/23 Well, first, dummy, 3780 is even, so it should be immediately clear that it's factorable...which is all you need to know when accessing if you should try factoring something. I suppose it might not have been clear that that factors the quadratic, but it puts you in the right direction.

[2] For future study: what if the coefficient of x^3 is not 1? What if there are 4 or 5 or 6 terms in the initial product?

2.11 Special Factorizations

Ideas:

i. Factorization of difference/sum of cubes is reversed from what one might expect without being careful. We can use factors to check factorization...ha. That's funny.

ii. If you have more than two (irrational root) terms, you can (sometimes) use difference of squares to get rid of some terms using clever combinations.

   Extra: When dealing with roots, you can probably use the conjugate if you don't know what to do. Look for diff of squares/cubes first, though.

Problems to revisit:

i. swag moola

ii. Another "find all pairs" problem

iii. The power of sum/diff of cubes/squares

iv. Conjugate works, but try using diff/sum of squares

v. solve *without* diff of square

vi. Good practice for making sure you've actually completed the problem

vii. swag

viii. Welp. It's definitely not 10 lol

ix. ....okay

x. This makes HMMT look easy lmaoo

2.12 Complex Numbers

Ideas:

i. Complex conjugates can be multipled by any real number other than 0 or 1 and will still have the "turn a complex number into imaginary" effect when multiplying...the other conjugate? See this for an example

ii. if w, z are complex numbers, then $\overline{w+z} = \overline{w} + \overline{z}$ and $\overline{wz} = \overline{w} \cdot \overline{z}$.

iii. $(a+bi)^3 = (a-bi)^3$. Where that will come in handy is beyond me, but whatever.

Quickly Revisit:

don't over think

is it EVER okay to divide in a problem with "find all"? (If you can do it with factor pairs instead of dividing, I'd try that first

2.13 Quadratic Equations pt. 2

Can derive quad formula by completing the square

Discriminant $b^2 - 4ac$ term in quad form. Determines nature of roots. discrimin ≥ 0 real roots. otherwise imaginary. clearly.

don't remember

Reaaaaaaaaaaaaaad. You are finding h, not sum of squares. You're told the sum of squares.

How can you find a way to represent the radical of a complex number as a complex number?

How can we turn two variables into one?

use variables to your advantage my guy

read; also, you need to revisit this until you can solve it

2.14 Graphing Quadratics

Ideas: negative signs screw things up

don't forget to flip flop

$ax^2 + bx + c = d$ is also the form of a prabola. You don't have to use standard form if standard form makes the problem gross.

Problems: circle ROUND

you never know what ideas are going to be helpful...geez that's a horrible name for this link...OOOGA BOOGA

ya gotta lawyer up

you really need to get better at reading lmfao

2.15 More Inequalites

Ideas:

i. Blind problems should be attempted without knowing the contents of the section.

ii. If you know a method that gauruntees it solves the problem but you don't want to use it, be very careful about wasting time on finding an "easier", and potentially nonexistent, solution method.

iii. Discriminant is ALWAYS useful when working with quadratics.

iv. Graphing is helpful, but at the right stages of a problem.

v. A quadratic greater than zero can never have real roots.

vi. We can work backwards to prove things. That is, if we are working with an inequality that we can manipulate into a true statement, we can (probably) prove a desired goal. Be careful about bogus solutions. To avoid false proofs, check your work by working forwards from where you ended while working backwards.

vii. will be useful for min/max of quad. Other than that, idk

viii. We use

$$\frac{x^2+y^2}{2 \geq xy}$$

in a very clever way here

Probelms:

i. solve without completing the square. If it's hard, you're doing it wrong.

ii. Blind problem

iii. Make sure you check for all sol!

iv. Close, but no cigar :(

v. What strategy can you employ to reduce error?

vi. More for Jeremy lmfao

vii. Blind/No Hint. If it's hard, you're doing it wrong!

viii. For shame

ix. Great practice applying trival inequality and AM-GM inequality

Questions:

1. Why can't we add x to both sides? Cuz multiple terms involved, our? Oh...maybe we can...

2. Can we use an inequality to compare real numbers and complex numbers? Probably not, right? Well, I guess real numbers are just complex numbers where the complex part is zero. So maybe. What about imaginary numbers? I think not.

3. Solution says we can't factor but I completed the square and got the same answer. Why is that invalid? Mmmm you sort of got the same answer. You also found something that could lead you astray. That is, you found that $x < \pm s$ where s is part of the solution. The problem with this is that it's not true. The actual solution is $-s < x < s$.

2.16 Functions

Ideas:

i. We can find domain by seting output to y and then factoring and such to find any such y where the function is undefined (dividing by zero, negative value equals square root, ineqalities etc.)

ii. Consider the function s where $s = f * g$ and $*$ is some operation (within reason). It should be noted that $s(a)$ is defined if and only if a is in the domain of both f and g. The safest way, then, to determine the domain of s is to *seperately* determine the domain of f and g and see where they overlap (and to check for any problems like dividing by zero).

iii. When working with composition, we need to be careful about range and domain. Consider some $f \circ g$. If the range of g is not in the domain of f, then $f \circ g$ will be not well defined for certain g.

iv. Notational note: We can denote multiple instances of composition with a power, much like higher order differentiation.

v. Be careful. If $g(f(x)) = 2x - 3$ and $f(x) = 2x - 3$, then prompt, "Find $g(4)$" is not asking for $g(f(4))$ because $f(4)$ is not 4. Rather, we need to find x such that $f(x) = 4$, then, plug said x into $g(f(x)$.

vi. Since f can only be an inverse of g if g is an inverse of f, if we know one, we can find the other using the fact that $f(g(x)) = g(f(x)) = x$. Which, to me, seems to be the same as the switching the x and y trick. However, there may be instances where switching doesn't work, so the composition trick is nice, too.

vii. If a function produces the same output for two different inputs, it does not have an inverse.

viii. To remember for Putnam Review: “Many complicated-looking functional equation problems can be solved with a little experimentation. Don’t let the notation scare you; these problems are often not nearly as hard as they look!”

ix. More to remember:

$$\text{Experiment} \rightarrow \text{Find Pattern} \rightarrow \text{Prove Pattern is True.}$$

Problems:

i. Restate or draw this in a way that’s more illuminating.

ii. Make a pretty picture that looks like a helix

iii. Bruh moment

iv. You need some food

v. Experiment and work backwards. If you really want a challenge, do it blind, without AoPS prompts:

The function f has the property that, whenever a, b, and n are positive integers such that $a + b = 2^n$, then $f(a) + f(b) = n^2$. Find $f(2002)$.

vi. you still need some food lmao

vii. You were close to solving algebraically and then you went into left field

viii. Bah! You should have had that one

ix. Some rewriting might help keep things organized

x. Some problems require words, not numbers

say waht? I genuinely don’t understand. It’s almost like we just made stuff up out of nowhere and somehow that’s not illegal.

Anyway...

2.17 Graphing

Did...did they get something wrong???

probably would be fine when you’re awake

2.18 Polynomials

If they don't cover the factoring/roots of higher degree polynomials in intermediate Algebra, come back to this problem and do the extra bit

It would seem I don't really understand composition

2.19 Exponents and Logarithms

Ideas:

- Substitution is a very powerful equation-solving tactic. Whenever you are faced with a complicated equation, consider substituting a variable for complex pieces of the equation. You might then be able to solve the resulting equation, and use that solution to solve the original equation.
- Way to rewrite logarithmic functions: If $f(x) = \ln(x)$, then $e^{f(x)} = x$.

Problems:

i. Blind

ii. The bruhest of bruh moments

iii. No peaking

iv. You missed a range and domain question about logs...bOi

v. Wowie

2.20 Special Functions

Ideas:

i. Domain/range of radical: input of radical must be nonnegative and output of radical must be nonnegative (so that it can be a function). From there, we can use inequalities.

ii. We can square equations to get rid of absolute value functions, if necessary. That is $|x|^2 = 2^2$ is the same as $x^2 = 4$ so $x = \pm 2$.

iii. When doing case work with absolute value functions, we should start with inequalities[3] and then form cases. We form said inequalities by considering bounds (variable values for which the absolute value function returns zero). See here for an example.

[3]Note: $\geq$ corresponds to net nonnegative input in abs val function and $<$ corresponds to negative

iv. We can find horizontal asymptotes by rearranging x in terms of y and seeing if there are values of y that cause a division by zero. We can find vertical asmyptotes where x causes division by zero. We care because **asymptotes are helpful in graphing rational functions**. Reminder: a rational function is just a **fraction where numerator and denominator are polynomials**.

v. Choosing values of x to elliminate a term makes partial fraction decomposition (and all sorts of other problems/concepts) much easier to manage.

vi. Just because I always forget, **partial fraction decomposition** ISN'T ACTUALLY HARD AT ALL. You simply factor the denominator, put each factor under a variable, and sum the terms, like so:

$$\frac{*numerator*}{*factor1**factor2**factor3*} = \frac{A}{f_1} + \frac{B}{f_2} + \frac{C}{f_3}.$$

Then, you simply multiply both sides by $f_1 f_2 f_3$ to get

$$numerator = Af_2f_3 + Bf_1f_3 + Cf_1f_2.$$

You can choose values for your input to make one term go to zero and then solve from there (with a systems of equations if necessary).

Problems:

i. Be careful with cubic roots, dummy!

ii. Good practice for avoiding sloppy arthimetic errors.

iii. You could try some inductive method, but there's a simpler way to achieve a similar result with inequalities. Can you find it?

iv. I know you're tired, but I mean cah maaaahn. You just learned this.

v. Be careful for multiple solutions!

vi. SWAG. No but actually see if you can prove the question at the end of their solution. Well the conjecture that corresponds to their question.

vii. Good practice being organized

Questions:

1. Is there some nice intuition we can use in this problem? In particular, what's happening with the center?

2. Does completing the square work for this problem?

2.21 Sequences and Series

Ideas:

i. We can rewrite terms in a sequence relative to a given term. In particular, if we have a finite sequence with odd terms, then we can rewrite everything in terms of the middle term, like so:

$$a-4d, a-3d, a-2d, a-d, a, a+d, a+2d, a+3d, a+4d.$$

This is particularly helpful in series problems. Notice, all the terms sum together such that the distance values cancel.

BE CAREFUL. You probably need to edit your notes. The middle method gives you the middle term, not the first.

ii. Consider the following question:

Suppose that the sequence $a_1, a_2, a_3, \ldots, a_{200}$ is an arithmetic sequence with $a_1 + a_2 + \cdots + a_{100} = 100$ and $a_{101} + a_{102} + \cdots + a_{200} = 200$. What is the value of $a_2 - a_1$?

Though I didn't, one should immediately recognize that $a_2 - a_1$ is just d, the distance between terms.

iii. The proof of Gauss's trick relies on clever arrangement. We can apply a similar trick to learn about geometric series. That is, consider what happens when we multiply a geometric series by its own ratio:

$$\begin{array}{rclcccccccccc} rS & = & & & ar & + & ar^2 & + & \cdots & + & ar^{n-1} & + & ar^n, \\ S & = & a & + & ar & + & ar^2 & + & \cdots & + & ar^{n-1}. & & \end{array}$$

Clearly, we can subtract S from rS and we will be very close to a desired formula for the sum of a geometric series.

iv. Some polynomials can be represented as a geometric series, which lends for very clever factoring tricks. See AoPS exerpt below:

The expression

$$1 + x + x^2 + x^3 + \cdots + x^{n-1}$$

is a geometric series with n terms, first term 1, and common ratio x. So, we have

$$1 + x + x^2 + x^3 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}.$$

Multiplying both sides by $x - 1$ gives us a factorization for $x^n - 1$:

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1).$$

Also, the whole fancy polynomial = geom. series trick is great for Putnam 2022 problem A2!

v. Be on the lookout for telescoping. It makes impossible problems easy! Telescoping is when terms can be grouped together such that everything but the first and last term (or few terms) cancel in a sum or product. See below:

$$(1000 - 998) + (998 - 996) + (996 - 994) + \cdots + (334 - 332) = ???$$

vs.

$$\begin{aligned} &1000 + (-998 + 998) + (-996 + 996) + (-994 + 994) \\ &+ \cdots + (-334 + 334) - 332 = 1000 - 332 \end{aligned}$$

vi. Some problems can be made simple by viewing the series as a sum of series. Notice that

$$\sum_{i=0}^{\infty} \frac{3^i + 5^i}{8^i}$$

can be rewritten as the sum of two different geometric series (one with ratio 3/8, the other with ration 5/8, both with first term of 1).

Problems:

i. Solve the sum using what you know about telescoping series. Hint: you may need to make some deductions before you can start solving.

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{99 \cdot 100}.$$

For the solution: see here. DO NOT PEAK UNTIL YOU HAVE WORKED FOR AT LEAST 10 MIN.

ii. Be careful. Adding and subtracting is hard lmfao. Try again

iii. You don't understand arithmetic sequences as much as as you think you do. When you review, you should go back and attempt the arthimetic sequence section again.

iv. YOU STILL CANT READ!

v. Really should have had that one

Questions:

1. Why can't we use the middle method here?

   MMMMM the middle method gives you the MIDDLE TERM.

2. Ayo, waht? Telescoping no go brrr. Ask Kirsten.
3. We got another, "Ayo, what?" situation over here. Ask Kirsten about how to handle seemingly unpredictable telescoping problems.
4. Ask Kirsten about cancelling variables and checking when something is or isn't okay.

2.22 Special Manipulations

Ideas:

i. Be on the lookout for conjugates and inverses. They can often lead to nice cancellations when an expression is squared.

ii. Symmetry makes systems of equations seem easy. That is, systems of equations that have symmetry can often be solved by combining all the equations at once.

iii. We've seen the idea I'm about to discuss before, but the problem below really helped me understand why it's so important. Write everything out and don't expect everything to simplify nicely. Sometimes things simplify when you recreate the information you already have. Notice, $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$ can be simplified by cubing it. It seems we get a mess, but what we get simplifies down to

$$x^3 = 4 - 3\left(\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}\right).$$

If we are not careful, we would think this means we have not made progress, but we have! Since the cube roots in the expression are what we started with, we have $x^3 = 4 - 3x \Rightarrow x^3 + 3x = 4 \Rightarrow x = 1$ so the entire cube root expression is just 1.

iv. Beautiful Problem

v. You need to stop dividing by a variable without first justifying that division. In this problem, they factored, which avoids any dividing mistakes entirely. I divided by a variable. In this case, that worked because we know $xy = 6$ so $x, y \neq 0$ which means division by x or y is fine, but I didn't think to check that while solving the problem. POOR FORM!

vi. Sometimes problems asking for numbers in a certain form are simplest to solve just by asking yourself, "What needs to happen for said form to occur?" For instance, if x is our desired solution to some problem and we know that solution has to be an integer, what must be true of a, b if $x = \sqrt{a+b}$?

vii. You had the right idea. Somewhere in your multiplication you mixed up terms which gave you the wrong expression. Naturally, it was hard to progress from there. I could tell that keeping terms straight with this problem was difficult. I should have rewritten things in a way that was easier for me. I should have slowed down. I should have been more organized.

Questions:

1. Ask Kirsten if we can use graph theory to simplify this toy problem.
2. https://artofproblemsolving.com/ebooks/intro-algebra-ebook/par/111765
3. My method worked, but I don't understand their method. Ask Kirsten.

Intro Algebra Finished on Feb 19 @ 10 pm

3 Intro to Counting and Probability

3.1 Counting Is Arithmetic

Skills and Concepts:

- Shifting a squence

  In problem 1.2.3 I recognized the sequence as an arithmetic sequence and used that to count the numbers in the sequence. However, in some cases that might make for nasty computation. Instead, we could recognize that we can shift every number in the sequence over and then divide by some factor to get something we know how to count.

- Categorical thinking

  Some problems involve overlapping categories. In those problems, we need to worry about over or under counting. The easiest way to avoid both is to take things one category at a time.

Problems

i. I need to have a more methodic way of handling scratchwork. This should have been easy.

ii. Don't overthink :)

Questions

1. Specific example/motivation of relevance?
2. Ask Kirsten if we can use induction on inequalities
3. Ask Kirsten about how to word why we can assume $x < y < z$ in this problem
4. Is there a way to use symmetry that makes geometric counting easier? Ask Kirsten

   squares

   triangles pt. 1

   triangles pt. 2 on same page

3.2 Basic Counting Techniques

Skills/Concepts:

- When doing casework, we should employ a strategic, organized method. For instance, in this problem, we count squares in a grid. To ensure we don't miss any cases, we use the Pythagorean theorem to confirm all diagonal squares we need to count.
- Overcounting is something we will see later, but it's already come up indirectly. In the mumble problem, we are asked to count the 3 letter words in which A is used at least once. One may be tempted to do 3×26^2, but such a method overcounts; notice it counts AAA 3 times—and probably other permutations as well.
- The phrases "at least" and "are not" are good indicators of problems in which complementary counting is probably a good problem solving strategy.
- Be careful about repitition. Some problems permit repitition while others do not. Make sure you don't assume either case until you have determined what the problem is really asking for.
- If it's not clear how to count what you want to count, consider counting what you don't want and subtracting it from all possible counts (complementary counting).

Problems:

i. Brenda and Ali got BEEF

ii. Ask Kirsten

iii. Challenge: solve without listing in any way (you may list in scratch work)

iv. Grand Pooh-Bah

v. Bruh

Questions:

1. Try a simpler case: poooop
2. Check with Kirsten cuz this was poo
3. Ask Kirsten about the 8 rooks problem

3.3 Correction for Overcounting

Ideas:

- When cause of overcounting is repitition. Two examples of occurences of repitition are permutations with repeated elements and permutations with symmetry. That is, notice that BAL_1L_2 has 4! permutations but $BALL$ does not. In each permutation of $BALL$, we count the permutation twice (once for L_2L_1 and once more for L_1L_2). Symmetry can be seen when counting permutations around a circle. To see the overcounting, imagine we cut the circle and flatten it out to form a line. If we rotate the circle, the corresponding line's permutation will still be the same.

- When correcting for repeated elements, DO NOT FORGET that if you group the repeated elements together, you are considering all the way the repeated elements AND THE OTHER ELEMENTS can be permuted.

Problems:

i. didn't get this one in pre-algebra either xD

ii. I need to revist all of section 3.4 (counting with symmetries). In particular, counting keys around a keychain is quite difficult for me.

iii. Another table problem

iv. Don't understand this. Might need to come back after learning more Geometry

v. Geometry and symmetry (regular tetrahedron)

vi. More symmetry (square)

3.4 Comittees and Combinations

Ideas:

- BINOMIAL COEFFICIENT: When order matters, there are $\frac{n!}{(n-k)!}$ ways to permute k objects from a set of n objects. However, when order doesn't matter, we are overcounting with the previous method. Notice, when order matters, k things can be permuted in $k!$ ways so we count every permutation above $k!$ times when we only want to count it once! Thus, we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \times (n-1) \times (n-2) \times \cdots \times (n-(k+1))}{k!}.$$

- Arithmetic sequences with k terms and a common difference of 1 can be written as $\binom{k}{2} = \frac{k(k-1)}{2}$ which might lead to a much simpler solution. Remember, n choose k is just a way of saying the number of ways we can form a k sized committee from a group of n people—which, is actually just another way of saying there are n ways to permute k objects from a set of n objects in which order doesn't matter ($BA = AB$).

- Just as we can think of n choose r to be the number of people on the committee, we can think of $\binom{n}{n-r}$ to be the number of people to *not* be on the committee. Therefore,

$$\binom{n}{r} = \binom{n}{n-r}.$$

  Notice, we can use n choose r to **save ourselves from messy computation**. What is $\binom{10}{8}$ equivalent to?

- Some problems require stepping back and reframing what it is we're trying to solve. For example, if we want distinct arrangements of $POOPOOP$ we need to consider where the Ps go. Notice, to get distinct arrangements, we really only care about how one of the two letters are arranged. Pick some arbitrary arrangement for the Ps. Then, notice that no matter how we interchange the Os, we get the same arrangement. Thus, the number of distinct arrangements of $POOPOOP$ is $\binom{7}{4} = \binom{7}{3} = 35$ where 7 choose 4 represents arranging the Os first and 7 choose 3 represents arranging the Ps first.

- **Problem Solving Strategy:** Don't get cocky just because you solved a few problems; you should be explaining your rationale using words, just like any other argument. Rather than stating $\binom{n}{k}$ you should be thinking about what n choose k represents in a given problem. That is, if you really must use notation, you need to justify that notation by providing an explanation for what the notation means in context.

Problems:

i. lol who taught you how to count? *joker reference*

ii. lololol slow down there broseuph

iii. See if Algebra actually works. Don't struggle too hard. If not, explain in words.

iv. Cmon, MAN! What's problem solving step number 1??? If you don't know how to solve it, consider what you would do in a problem you do know how to solve!

v. You need to stop assuming it's as simple as bleh bleh choose deh deh. Remember, this is a counting book, not a plug, chug, and choose book.

vi. If it's hard to count what you want, count what you don't want!

vii. Ask Kirsten lmfao counting cards is hard

I think I sort of understand. We want a five card hand. 3 of the cards need to be of the same rank, but we don't care what suit they are, and more importantly, we don't care what order they are in, so it's a combination, not a permutation. The last two cards must have a different rank from each other and the other 3 cards, but we will deal with those later. First, we tackle the same rank cards. Since we don't care about the order of the suits—jack club spades is same as spades club jack and so on—we should use a combination. We could view this as 4 choose 3, but I think it's more intuitive to think of all the ways we could *not* choose one suit. Namely, there are four suits we could choose to omit. Once we've ommmitted a suit, the three remaining suits have 13 different ranks that could act as our 3 card rank. Thus, the ways to combine 3 cards with the same rank is $4 \cdot 13$. The next card must be a different rank, so there are 12 viable ranks. We don't care about the suit, so the possible choices for the next card are $12 \cdot 4$. A similar idea applies to the next card. The thing I don't understand is the correction for overcounting. Anyway.

Questions:

1. Could we use combinations to simplify some of the geometric problems from earlier in the book? In particular, could we solve the rectangles in the grid problem with a combination? Mmmm maybe not, because the "order" matters. If we choose *any* four points, we could get a line instead of a rectangle...

3.5 More with Combinations

Ideas:

- In this probelm we wish to split seven men and three women up into two groups such that each group has at least one woman. If we try to do everything at once, it would be difficult. If, instead, we split the problem into sub-problems and focus on how many ways there are to split up women and how many ways there are to split up men the task will be much easier. Divide and conquer!

- Unfortunately, the summary for this chapter is not very good. You definitely need to revisit sections 5.3 and 5.4. Revisiting chapter four is probably a good idea, too.

Problems:

i. You done bamboozled yo self

part 2 lmao

ii. bruh moment

iii. ask Kirsten and try again another day Why is it 10 choose 3? Couldn't that result in instances where the first digit is zero? Ha! no. Notice that 0 must be in units place in every ordering. Choose three random numbers in the set {01...9}. Notice that for any combination of 3, the descending order is $x_ix_jx_k$ where $x_i > x_j > x_k$. Also, I probably need to start a category in which I explain answers to myself.

iv. Sometimes it's undercounting, not overcounting, that we need to worry about

v. Distinguishability

   Do this one, too

vi. What's 9 + 10?

vii. :(sadge

viii. Spent all that time just to get it wrong

ix. I think this should be in a different section but oh well

3.5.1 Farkle

Question: What's the probability that you bust after you're around the corner in farkle?

Solution: In any probability problem, it's usually good to start by defining your sample space (counting all possible outcomes). Suppose that our six dice are colored so that they are distinguishable. It then follows that there are 6^6 possible outcomes.

We now use complementary counting to determine the number of ways in which we can bust. I claim we have roughly 11 cases to consider (some of which have sub cases).

Case 1: 1 through 6. Not counting for ordering, there is exactly 1 way to roll 1-2-3-4-5-6. However, since our dice are distinguishable, there are 6! ways to roll 1 through 6.

Case 2: Pairs. There are $\binom{6}{3}$ ways to get 3 pairs. Once those pairs have been selected, we need to order the pairs. We now need to be careful. Do not make this mistake of assuming the pairs need to be paired. That is, 1-1-2-2-3-3 is a roll with pairs, but so is 1-2-3-3-2-1. We can think about all possible orderings of 3 pairs by recalling our method of counting permutations with repeated elements. Namely, to count all the arrangements of the letters F, o, o, d we pretend the o's are distinguishable by using subscripts, and then we correct for overcounting. Namely, if we pretend the o's are o_1 and o_2, there are 2 ways to arrange them so there are $4!/2$ ways to arrange F, o, o, d. Hence, there are $6!/2^3$ ways to arrange the pairs. Therefore, we have $\binom{6}{3} \cdot 6!/2^3$ ways to get pairs.

Case 3: Double Triplets. A double triple occurs when you have two three of a kind in the same roll (e.g. 1-1-1-2-2-2, 3-4-3-4-3-4, etc.). There are $\binom{6}{2}$ ways to get the the two numbers that will form our three of a kind. Then, using a similar argument to Case 2, there are $\frac{6!}{3!^2}$ ways to order the triplets. Thus, there are

$$\binom{6}{2} \cdot \frac{6!}{3!^2}$$

ways to get two triplets.

Case 4: 6 of a kind. There are exactly 6 ways to get 6 of a kind.

Case 5: 5 of a kind. There are 6 choices for the number to form the 5 of a kind. Then there are 5 choices for the remaining die. They can be ordered in 6 ways (slide the one outlier over 1 slot 6 times). Hence, there are $6^2 \cdot 5$ ways to get 5 of a kind.

Case 6: 4 of a kind. Unfortunately, we need to break this into sub cases. Suppose that we get 4 of a kind and then 2 of a kind. There are 30 ways to choose the two numbers, and then there are $6!/(4! \cdot 2!) = 15$ ways to order the roll. Then, suppose that we get 4 of a kind and then 2 distinct dice. There are $6 \cdot 5 \cdot 4$ ways to choose the numbers, then there are $6!/4!$ ways to order the roll. Hence, there are

$$30 \cdot 15 + 30 \cdot 120 = 30 \cdot 135$$

ways to roll 4 of a kind.

Case 7: 3 of a kind. Since we are considering 3 of a kind to be distinct from double triplets, we can either get 3 of a kind and 3 distinct dice, or we can get 3 of a kind, two of a kind, and one of a kind. That is, we could get, as an example, something of a similar form to 1-1-1-2-3-4, or something of a similar form to 1-1-1-2-2-3. The first sub case can be done in $6!/2$ ways and ordered in $6!/3!$ ways whereas the second subcase can be done in $6!/3!$ ways and ordered in $6!/(3! \cdot 2!)$ ways. Hence, there are

$$\frac{6!}{2} \cdot \frac{6!}{3!} + \frac{6!}{3!} \cdot \frac{6!}{3! \cdot 2!}$$

ways to get 3 of a kind.

3.6 Some Harder Counting Problems

Ideas:

- During one of the AoPS videos, Richard demonstrates excellent organization. With proper organization, a hard problem became a do-able, or even easy, problem. It's not a perfect similie, but I'd say organization is to counting as notation is to algebra. If you choose nice notation and write everything out, the ideas are easy to conceptualize.

- Geometry is *extremely powerful*. Just as we recognized that we can use basic geometric in the not-cycloid cycloid Putnam problem to avoid integration, we can use geometry to form valid combinations where there didn't seem to be any at first.

- Being stubborn $\neq$ being persistent

  "Be flexible with your problem solving approaches. If you try an approach and it doesn't look like it's going to work, don't be stubborn–think about starting over with a new approach."

- Noticing that complementary counting is possible is not always obvious. If you don't know how to count what you want, like before, start with the no restrictions case and work backwards. If it isn't obvious as to how to count the restriction, ask yourself if there is a way to remover items from a list of *all* permutations. That is, instead of counting (adding), do some "anti-counting" (subtracting).

- Representing abstract ideas as words with repeated elements (if necessary) is a very powerful tool. It's easy to count words. For example, we can easily count the paths from one lattice point to another in n dimensional space using the word idea.

Problems:

i. I'm going to put this problem in the AoPS geo problems section, as well. It's a straightforward problem, but it requires good geometric awareness to solve quickly.

ii. You're not really at the "check your answer" stage for Putnam problems, but this problem is a great example of why you do need to check your answer when you can.

iii. Be careful about indexing!

iv. Solving the problem with multiplication, though possible, is a royal pain. Try to be clever and use addition.

v. Right Idea, Wrong Implementation

vi. Tunnel vision go brrr

vii. Clay targets

   Do the rest of the problem above as well.

viii. Lol checking numbers to see if the number is even remotely reasonable would help.

Uhhh the heck is this problem?

3.7 Intro to Probability

Ideas:

- Be consistent in your counting method from problem to problem when working on probability. That is, if it is easier to use combinations in one problem and easier to use permutations in another, do it! But be sure that you don't overcount; if you consider order for all outcomes, you must consider order for successful outcomes as well.
- Some problems just require casework, careful arithmetic and a lot of patience—blocks with different materials, sizes, shapes, and colors, I'm looking at you!.
- Be careful about assuming indepdence. Say some sports team has a probability p to win and their star has a probability s to score. Notice, if the star scores, winning is more likely whereas if the star doesn't score, winning is less likely, so the events are *not* independent.

Problems:

i. Should be quick

ii. bruh moment

iii. lol you can't count boi Dude. You compared apples to oranges!

iv. Nice

v. I've seen this problem in one of the previous AoPS books (I think Pre-Algebra). At that time, I didn't stand a chance. This time around, I almost got it. I tried a technique, eventually concluded it wouldn't work, moved on, discovered I would need to do six cases, didn't know how to solve the cases so kept experimenting, finally figured out how to solve the cases, and threw in the towel once I solved it conceptually.

3.8 Basic Probability Techniques

Ideas:

- When two events are mutually exclusive (one or the other can occur, but not both), then we simply add their probabilities together to find the probability that one or the other occurs. If two events are *not* mutually exclusive, we must be careful about overcounting.
- Your brain should immediately light up the "be careful" alarm when phrases such as "**at least**" or "**at most**" are in a problem.

- I've come up with this concept on my own, so I can't use it as if it were a theorem (unless I prove it, and then it's fine). I've noticed that we can do the "outcome method" or the "liklihood" method. In problems where probability is defined for us and events are not equally likely, I think we need to use the liklihood method (multiplying probability of independent events). There are some cases where both methods work. For example, the probability that we roll a 1 or a 2 in 3 out of 5 rolls can be found in (at least) two ways:

  1. $P(1 \textit{ or } 2) = 2/6 = 1/3$ and $P(!(1 \textit{ or } 2)) = 4/6 = 2/3$ and there are $\binom{5}{3}$ ways to get 3 out of the 5 rolls to be rolls we want. Thus, the probability is
  $$\binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^5} + \cdots + \frac{2^2}{3^5}$$
  where $\frac{2^2}{3^5}$ occurs 10 times (because 5 choose 3 is 10). Notice, the two rolls SRRSS and RSRSS—where R denotes a roll we don't want and S denotes a roll we do want—cannot happen at the same time, so we can add their probability (mutually exclusive events).
  2. Use the outcome method: 6^5 possible outcomes, 2 outcomes that we want per roll and we want 3 rolls, 4 outcomes that we don't want and for this we want two rolls, and, like before, there are 5 choose 3 ways to get the 3 rolls we want. Thus, we have
  $$\binom{5}{3} \cdot \frac{2^3 4^2}{6^5}.$$
  Notice that the fraction above is equivalent to $\dfrac{2^7}{2^5 3^5} = \dfrac{2^2}{3^5}$ so we get the same result we found earlier.

Problems:

i. 5 white balls and k black balls are placed into a bin. Two of the balls are drawn at random. The probability that one of the drawn balls is white and the other is black is $\frac{10}{21}$. Find k.

ii. We flip a fair coin 10 times. What is the probability that we get heads in exactly 8 of the 10 flips?

iii. Three cards are dealt at random from a standard deck of 52 cards. What is the probability that the first card is a 4, the second card is a ♣, and the third card is a 2? I had right idea, wrong implementation.

iv. My school's Future Mathematicians of America club has 16 members, 7 boys and 9 girls. A president and a 3-person executive committee are chosen (where the president cannot serve on the committee). What is the probability that the president is the same gender as the majority of the committee?

v. Probability? More like Bobability *crickets chirp*

vi. Wayne and Mario play a game in which they take turns flipping a fair coin. The first one to flip tails wins. Wayne goes first. What is the probability that Wayne wins?

If a problem doesn't say "in terms of", you should try to find a number. If you found something in terms of a variable and it was really easy, you should DEFINITELY try to find a number.

vii. In the card game bridge, each of 4 players is dealt a hand of 13 of the 52 cards. What is the probability that each player receives exactly one Ace?

There are several ways to solve this problem. Find the (computationally) easy way.

viii. I got the right answer, plugged it into desmos even though they sort of told me not to, couldn't believe it, remembered the birthday paradox video, still thought my answer was wrong for some reason, and then did some weird summation sum trying to find the answer without complementary probability. If I'm really bored and have some free time (lol that's funny), I could try to do the summation version again some day.

ix. I don't understand sports betting odds. Oh wait. Maybe I do. Lol I think I do. Damn I'm dumb. How did I not get that. I think if the odds are x-to-1 then the probability that the horse or the team or whatever wins is x times more likely that the horse/team/whatever loses. We know that P(win or lose) = 1 so xP(lose) + P(lose) = $1 \Rightarrow$ P(lose) = $\frac{1}{x+1}$. I think? I don't know. It's still confusing.

Questions:

1. What's the difference between two events that are independent and two events that are mutually exclusive? In the text, we add for ME and multiply for independent. Ahhh aha! We used addition for $P(A)$ **or** $P(B)$ whereas we use multiplication for $P(A)$ **and** $P(B)$. It's also worth noting that earlier in the section the probabilities we were working with had the "equally likely" property whereas some of the probabilities we're working with now do not.

   The value of one outcome does not influence the probability of the other outcomes. That is, unlike mutually exclusive events two independent events can happen at the same time.

2. I still don't really understand how to identify when I should be multiplying and when I should be adding.

3. I also don't understand the symmetry of coin flipping.

4. How can we do this problem without using the fact that even and odd are equally likely?

5. Probability of a king then a heart is the same as probability of king and heart. Try thinking about and vs or. adding and multiplying being the same in right circumstances. Swag

3.9 Think about it!

Ideas:

- Though we must be very, very careful when employing this tactic, we can sometimes ignore irrelevant information.

Problems:

i. Solve the problem without using calculation or equations.

ii. Joe's Coffeehouse? Nah Joe's Pizza

iii. See if you can solve this problem faster after working through the number theory text.

iv. Chess Match

v. Not really representative of the problems I'm interested in, but fun problem all the same.

vi. Very fun problem if you can figure it out

vii. The Last Boarding Pass Problem

Questions:

1. Doesn't the solution to this problem depend on 37 being placed first?

2. These symmetry tricks seem very gimmicky. I don't find them particularly illuminating, but I do see why they would be useful if I could actually understand them.

3. Again, I don't really see how we get Annika's probability of being first.

4. it's TWO OUT OF 13???

5. I genuinely don't believe this is true

3.10 Geometric Probability

Ideas:

- We can use geometric probability in problems where outcomes are continuous instead of discrete (try and see if the shooting stars problem can be solved with GP!).
- Charts and intervals are great tools we can use to stay organized.

Problems:

i. Read more carefully or go to sleep!

ii. Not sure how I messed that one up.

iii. My friend and I are hoping to meet for lunch. We will each arrive at our favorite restaurant at a random time between noon and 1 p.m., stay for 15 minutes, then leave. We want to determine the probability that we will meet each other while at the restaurant. (For example, if I show up at 12:10 and my friend shows up at 12:15, then we'll meet; on the other hand, if I show up at 12:50 and my friend shows up at 12:20, then we'll miss each other.) This is a fun one.

iv. Three points x, y, z are chosen at random on the unit interval $(0, 1)$. What is the probability that $x \leq y \leq z$?

   How many times do I have to tell you??? If they don't say "in terms of" then you should give a number!

v. Triangles, Area, and Probability

vi. Choosing a point at random inside a square.

3.11 Expected Value

Ideas:

- We can use the expected value to find the fair price of a game. A fair price is a price such that, on average, no one makes or loses any money. If we play the game for a 1000 times in a row, we should expect the value at the end to be zero. Thus, if we a designing a carnival game in which a player pays us to play, has some odds of winning w, then we calculate the expected value, we raise our price a little, and then on average we should profit.
- Could make for a very interesting introduction to infinite geometric series

- Expected Value is great for solving problems that are impossible to count manually. Examples of such problems are

  1. The number of flips necessary to get heads on an unfair coin.
  2. The number of flips expected to occur in a game in which two players exchange objects.
  3. Fair price problems and profitability.
  4. Probably some other stuffs.

  In general, though, notice that the problems I mentioned above care about ***one*** one number as the end result.

- Be on the lookout for geometric series in disguise in expected value problems.

- We can sometimes define expected value in multiple ways. For instance, in problems with iteration, we might be able to define the expected value at the nth iteration or we can define the expected value by looking at the beginning and the end of the problem. If the problem is of the right form, there will be a relationships between the two different versions of expected value we found, which we can then use to solve the problem.

Problems:

i. Right idea, wrong implementation

ii. oops Do rest of problem too

iii. Good Problem; helps if you don't completely mess up your definitions xD

iv. A valuable treasure worth $100,000 fell out of my airplane and landed in a large swamp. If I don't find it within a day, it will sink to the bottom and never be found again. I can hire one or more helicopters to search the swamp. Each helicopter costs $1,000 to hire for a day, and has a 90% chance of finding my treasure. How many helicopters should I hire? Don't forget your axioms of probability!

v. Magic with expected valu In general, you should do most or even all of the challenge problems again sometime.

Questions:

1. I cannot answer the extra question in their solution to this problem. Or at least, I can't answer it on an empty tank.

2. What's the explanation they refer to for the switching envelope paradox?

3. Are we supposed to know about the complement of probability? I suppose not but the probability that either of them find it *is* the probability of the "the complement" of neither of them finding it, which is really small. That is, it's unlikely that neither of them find it, so it's likely that one of them finds it.

   This is magic. I don't believe in this. How does this actually work?

3.12 Pascal's Triangle

Ideas:

- **Combinatorial Proof:** As we progress further in mathematics, we will find there are many ways to solve a problem. One of the tools we can now add is the concept of combinatorial proof. Namely, we can observe that we could prove Pascal's Identity algebraically, or, we can use combinations. Notice that if we have n marbles and we wish to choose r of them, there are $\binom{n}{r}$ ways of doing that. Similarly, consider an indivual marble m from the n marbles. Notice that the marble is either in a given combination or it isn't. Thus, the number of ways to choose r marbles from n is given by the number of ways marble m can be in one of the combinations plus the number of ways m can't be in a combination. Hence, we have

  $$\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$$

  where the first combination comes from the cases where m is in the combination (so we choose the remaining $r-1$ marbles from $n-1$ marbles) and the second combination comes from the cases where m isn't in any of the combinations (so we choose r marbles from the remaining $n-1$ marbles).

  Note: the identity above is conventionally known as *Pascal's Identity*.

- **Combinatorial Identities**

  1. *The Sub-committee Identity:*
     Consider the identity
     $$\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r}.$$
     We can think about this problem by thinking about a club with n memebers which wishes to form a committee with m members and a sub-committee of the committee with r members. There are two ways to do that. We could form the committee first, then form the sub-committe, or we could form the sub-committee and then form the remaining part of the main committee. Both methods will achieve the same number of combinations.

2. *The Men and Women Identity:*

   If we have a group of m men and w women, then the number of ways to form a committee from that group is clearly $\binom{w+m}{r}$. Alternatively, we can notice that every committe will have a certain number of men or women. For instance, there are $\binom{m}{0}\binom{w}{r}$ ways to form a committe of only women. Likewise there are $\binom{m}{1}\binom{w}{r-1}$ ways to choose a committee with exactly one man. Continuing on this way, if we combine all committees with exactly $r-k$ men where $0 \leq k \leq r$ we will have formed all possible committees, which we already showed is $\binom{m+w}{r}$. Therefore, we have the identity

   $$\binom{w+m}{r} = \binom{m}{0}\binom{w}{r} + \binom{m}{1}\binom{w}{r-1} + \cdots + \binom{m}{r-1}\binom{w}{1} + \binom{m}{r}\binom{w}{0}.$$

3. *The Men and Women Identity in Disguise*

   Consider the following identity:

   $$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n} \tag{1}$$

   Suppose that in the last identity, $m = w = n$ so there are n women and n men. Then, like before, we find the number of committees with no women, one woman, two women, and so on until we have the number of ways to form a committee of n women. That is, we have

   $$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0} = \binom{2n}{n} \tag{2}$$

   Since $\binom{n}{k} = \binom{n}{n-k}$, it is clear that (1) and (2) are equivalent.

- **Using Identities to Prove other Identities:** Sometimes we can employ cheeky tricks to solve problems that would otherwise be difficult. For instance, consider the identity

  $$\binom{2n}{n} + \binom{2n}{n-1} = \frac{1}{2}\binom{2n+2}{n+1}.$$

  We can use Pascal's Identity to prove the above. I won't type out the proof, because I'm a lazy bastard, but I will link it here. Alternatively, you may rederive it on the spot. It's not that difficult so long as you remain organized and think creatively.

  Edit: It would seem that Pascal's Identity is particularly helpful. I've seen it come up in various different proofs now. A very elegant one proves $\binom{2n}{n}$ is always even. Notice, we can use Pascal's Identity to write

  $$\binom{2n}{n} = \binom{2n-1}{n-1} + \binom{2n-1}{n} \Rightarrow \binom{2n}{n} = 2\binom{2n-1}{n-1}$$

where the implication comes from the fact that $(2n-1)-n=n-1$ and we know

$$\binom{c}{r} = \binom{c}{c-r}$$

so the proof is complete.

- **An alternative proof to the cardinality of a finite subset:** The text used walking paths to prove that

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

We will consider sets and subsets instead. Notice, if we wish to form all subsets of a set S with n elements, we count all the ways we can form a subset with...no elements, with one element, with two elements, and so on up to n elements. Notice also that for each element in the original set S of n elements, there are two choices pertaining to that element and any given subset of S, namely, the element could be in the given subset, or not. Thus, there are 2^n ways to form a subset from a set with n elements. Since the sum we found early is the number of subsets of S we have

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

as desired. In this proof we used a very powerful idea. Namely, we discovered that **we can often prove combinatorial identities by counting the same quantity in two different ways.**

- Pascal's Triangle has many, many patterns. For example, the Fibonacci numbers are hidden in Pascal's Triangle. When we sum every other number in row n, we get 2^{n-1}. And on and on...

- Sometimes we just need to get our hands messy and do some algebra...

Problems:

i. Combinatorial proof go brrrr

ii. You did an algebraic proof while attempting the combinatorial proof. Try combinatorial again.

iii. Combinatorial proof is a lot of fun...if you understand it lmfao

iv. Presidents and committees = combinatorial proof

v. What happens when we add every other term in each row of Pascal's Triangle?

vi. You apparently have forgotten how to work with ratios :(

Questions:

1. The solution to this problem relies on pairs, but I fail to see how we can pair things if n is even as there will be an odd number of combinations.

3.13 The Hockey Stick Identity

Ideas/Problems:

- A complicated process giving a simple answer often means that there is a simpler explanation for the simple answer.
- NEED TO SUMMARIZE HOW WE SIMPLIFIED PROBLEM TO SOLVE PROBLEM
- Delimeters can make hard counting problems...less hard.
- We can handle restrictions with clever casework. Can you see how?
- Some restrictions suggest that we should change the problem entirely. For example, one problem asks us to count the number of different vote counts if 3 students in a class of 70 students are running for president and some student(s) don't vote. How can we keep track of the votes that aren't submitted? Hint: How do we keep track of the votes that *are* submitted?
- Sometimes we must reword problem in order to recognize how to solve it. Can you see how we could reword the following problem? In how many ways can we distribute 7 pieces of taffy and 8 pieces of licorice to 5 kids, such that each kid receives exactly 3 pieces of candy?

3.14 The Binomial Theorem

Ideas:

- Some problems can be solved by plugging in nice values into to variables to make them go away. Notice, we can prove the cardinality of a finite power set very easily with the binomial theorem:

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n}y^n \\ \Rightarrow (1+1)^n &= \binom{n}{0}(1)(1) + \binom{n}{1}(1)(1) + \cdots + \binom{n}{n}(1)(1) \\ \Rightarrow 2^n &= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.\end{aligned}$$

- The binomial theorem can be understood easily with combinations. Notice that when we expand $(x+y)^n$, we will have every possbile combination of x and y. That is, we have $(x+y)(x+y)(x+y)\cdots(x+y)$ where there are n binomials and we must have every combination where we choose a y from 0 binomials, one binomial, two binomials, three binomials, and so on up to n binomials. Notice, when we choose k y from the n binomials, we can do this in $\binom{n}{k}$ ways and every term will have the form $x^{n-k}y^k$. All told, we get

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n.$$

- We were not given proof, so I wouldn't use it in any setting that doesn't accept references, but apparently the binomial theorem works with noninteger powers. That is, for $x, y, r \in \mathbb{R}$ and $|x| > |y|$ we have

$$(x+y)^r = x^r + rx^{r-1}y + \frac{r(r-1)}{2!}x^{r-2}y^2 + \frac{r(r-1)(r-2)}{3!}x^{r-3}y^3 + \cdots$$

One use of the above is to approximate radicals. For instance, we can approximate $\sqrt{5}$ by writing out the binomial expansion of $(x+1)^{1/2}$ and substituting $x=4$.

- There are some problems in which the terms suggest at the use of the binomial theorem, but it cannot be immediately applied. For example, one problem asks us to compute the sum

$$\binom{20}{20} + \binom{20}{18}\left(\frac{1}{2}\right)^2 + \binom{20}{16}\left(\frac{1}{2}\right)^4 + \cdots + \binom{20}{0}\left(\frac{1}{2}\right)^{20}.$$

Notice that if we let some variable v_1 be given by the sum above, and we let some other variable v_2 be the remaining part of the binomial coefficients and their corresponding variable terms that we would normally expect to find, then we know what $v_1 + v_2$ and $v_1 - v_2$ are by plugging into the variables that correspond to their binomial expansions. From there, we can do some very clever trickery. That is, we can simply add and subtract the same thing:

$$v_1 = \frac{1}{2}\big(v_1 + v_2 + v_1 - v_2\big) = \frac{1}{2}(2v_1) = v_1.$$

Since we know the value of $v_1 + v_2$ and $v_1 - v_2$, we get an answer.

Edit: That's not a very easy trick to remember, but fortunately, I don't think we have to remember the whole trick. The important part is to remember that if we have an expression with a bunch of combinations that seem to iterate, that may call for the binomial theorem.

Problems:

i. Behold, the messy algebra!

ii. Very cleva!

iii. Use counting to find the coefficient of the $x^iy^jz^k$ term in the expansion of $(x+y+z)^n$. Write your answer in terms of n, x, y, and z.

iv. Good problem, bad timing lolololol

Questions:

Is this the pyramid fractal thingy on C's desk?

3.15 More Challenging Problems

There are 6 problems in the regular section of this chapter. One of those problems is a USAMO problem which I obviously went to first. The work that I completed was correct, but I did not finish the problem. I find this to be a recurring pattern in my attempts at challenging problems. I believe I should experiment with a new method. Namely, if I know I'm going to be working on a few challenging problems, do what I can with one problem, when I get stuck enough to give up, move on to the next problem and eventually circle back to see if a little time away from the problem has helped me out. I will try this method on the remaining challenging problems. It's currently 4:36. I'm going to take a four minute break, start at 4:40 and give myself 30 minutes. No peaking until after the thirty minutes have passed.

Results:

I'm quite frustrated to say I only got one correct answer. I was close for several other problems, but close isn't good enough.

3.15.1 Tennis Tournament w/ 64 Players

I got this correct.

3.15.2 Chemistry Class

I miss counted. I did 1/(15 choose 3) but I forgot that the other groups can be mixed up so there is more than one correct outcome out of the 15 choose 3 possible outcomes.

3.15.3 Olympiad Problem

Notes are above and in calendar (March 5th).

3.15.4 Pascal's Triangle

Did not complete.

...And I should have. I completely misread the problem.

3.15.5 AIME Rhombicuboctahedron

I am extremely pleased about what I was able to solve in this problem. I'm happy I pressed forward and got a solution. I'm quite frustrated my solution was wrong. I failed to consider complementary counting, and if I had, I might have been able to solve the problem. With the complementary counting approach, we count edges and interior angles. We know how many squares, hexagons, and octagons there are, so we can count both edges and interior diagonals for each face. Note: we use the $n(n-3)/2$ rule to count interior diagonals. From there, we recognize that each edge shares a face (though it isn't entirely obvious to me that that would be known beforehand) so we need to divide our edge count by two. From there, we know there are 48 choose 2 total segments so we subtract edges and interior diagonals to get our desired result.

The other method, which is what I tried, is, in my opinion, not really possible without either some luck or knowledge of the rhombicuboctahedron. Somehow, one needs to know which verticies are being overcounted. I tried to deduce that, and failed. Though, truthfully, after some reflection, I was close. I should have tried to confirm with another method. My thinking was actually not as bad as I thought it was, but I miscounted the verticies that are "bad". If I had simply drawn out what I thought I had deduced, I would have solved the problem correctly.

3.15.6 Twins and Tennis

I recognized that this problem was just a fuckton of casework, and I refuse to do that in my tiny journal. I had a decent approach. DNF. Oh well. It's probably worth looking at again sometime, though.

3.15.7 9 sided polygon and probability

I do not understand the solution to this problem.

Problems to re do:

Just re do all of them lol

1. Don't confuse lists and permutations...
2. Describe integers n for which the expansion of the given expression is a constant.
3. Rose writes all the 7-digit numbers in which all the digits are different and each digit is greater than the one to its right (so the tens digit is greater than the units, the hundreds greater than the tens, and so on). And then some stuff happens!
4. I have a bag with 5 quarters and 3 nickels. I randomly draw coins out one at a time. What is the probability that I haven't removed all 3 nickels after 4 draws? Find a clean solution with very little computation.
5. A piece of string is cut into two pieces at a point selected at random. What is the probability that the length of the longer piece is at least 4 times the length of the shorter piece?
6. Three fair 6-sided dice are tossed. What is the probability that the sum of the numbers on two of the dice equals the third? Not asking to provide illumination...can we find a reason for the nice solution? It seems too nice to be a coincidence.

Finish time is grey area. March 6 kind of, March 20 kind of

4 Intro to Number Theory

Started March 6, 2023

4.1 Integers: The Basics

To my surprise, there is actually something (sort of) new in this first chapter.

Ideas:

- Be as restricting as possible. If we are told that the average of four numbers is an integer and that three of the numbers are 94, 91, 95, and one of the numbers is between 81 and 87 inclusive, then we can simplify the problem by considering the fraction
$$\frac{95 + 94 + 91 + (81 + n)}{4}$$
with n between 0 and 6 inclusive as opposed to
$$\frac{95 + 94 + 91 + x}{4}$$

with x between 81 and 87 inclusive. I suppose my method would have been effectively the same, but I think their's is more elegant.

- A perfect power is a power of *at least* 2, **not** exactly 2.

- Some number theory problems are actually counting problems.

- Inequalities can turn a weak argument into a proof when used properly. Use them!

4.2 Primes and Composites

Ideas:

- The Sieve of Nygard

  Let's say we want to determine if 2317 is prime. Notice, 2310 is $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. Then, we can quickly elliminate possibilites by counting multiples of 2,3,5,7,11 before and after 2310. When we do, we do not see 2309 or 2311 but we do see 2317. So, 2317 is *not* prime, but 2309 and 2311 might be (and actually, they are).

Problems:

i. Is 9409 prime? If you need a hint, see this footnote[4].

Questions:

1. I don't entirely understand how Nygard's Sieve will help us here unless we already know a really nice number.

4.3 Multiples and Divisors

Ideas:

- The division algorithm is a surprisingly powerful tool.

- ...I don't really know how to express this idea other than to say modular patterns are something I need to strengthen. In one problem, we are asked to find the number of integers between 100 and 700 that have a remainder of 1 when divided by 5. The solution used the division algorithm, but it also pointed out that we can represent

[4]What method could we use to very quickly acquire upper or lower bounds for the perfect squares we're looking for?

the integers between 100 and 700 as a series of rows of 10. That is, we would have 60 rows, there are two numbers per row that satisfy our desired specification—because $1 \mod 5 \equiv 6 \mod 5$—so there are $60 \cdot 2$ integers that leave a remainder of 1 when divided by 5 in between 100 and 700.

Questions:

1. Why is the solution to this problem so pretty? There seems to be some modular trickery going on. Or maybe I'm just dumb and don't understand numbers. Mmm I think I'm just dumb. Notice, 42 and 77 don't change throughout the problem.

4.4 Prime Factorization

Ideas:

- Though the power of prime factorization is nothing new, it should be noted that sometimes subtle details show up in unexpected ways. For example, we know perfect squares have a prime factorization such that all powers are even. Thus, if we are asked to find the five smallest positive multiples of 8 that are perfect squares, we start by recognizing the even power property, then we recognize that multiples of 8 that are perfect squares have the form $(2^2 \cdot n)^2$ where $n \in \mathbb{Z}_+$.

Problems:

i. How many possible values are there for the sum $a + b + c$ if a, b, and c are positive integers and $abc = 72$?

ii. The product of any two of the positive integers 30, 72, and N is divisible by the third. What is the smallest possible value of N?

iii. It might be time for bed...

4.5 Divisor Problems

Ideas:

- One would hope that I'd be more organized by now, but I'm not. When counting divisors, it is critical to make a chart. Observe the chart that we use to find the divisors of 200. We find the prime factors, and then we do all possible combinations of factors.

$a \backslash b$	0	1	2
0	$2^0 \cdot 5^0 = 1$	$2^0 \cdot 5^1 = 5$	$2^0 \cdot 5^2 = 25$
1	$2^1 \cdot 5^0 = 2$	$2^1 \cdot 5^1 = 10$	$2^1 \cdot 5^2 = 50$
2	$2^2 \cdot 5^0 = 4$	$2^2 \cdot 5^1 = 20$	$2^2 \cdot 5^2 = 100$
3	$2^3 \cdot 5^0 = 8$	$2^3 \cdot 5^1 = 40$	$2^3 \cdot 5^2 = 200$

- Perfect squares have an odd number of divisors.
- The common divisors of 2 integers is the divisors of their GCD. Conjecture: the same idea holds for n integers. Can you prove it?
- Yet again I find myself underappreciating the act of organizing one's work. If we are trying to understand divisors, it's helpful to write out the form of the divisor. That is, if we want to know something about the divisors of a number n with a prime factorization $2^3 3^4 5^6$, we should start by writing out $d = 2^a 3^b 5^c$. This way, we don't lose track of possibilities and we have something to work with moving forward.
- Rewriting equations backwards is ultra giga big brain. Okay, but serisouly, we've written equations backwards to find Gauss's trick, solve all sorts of different series problems, and now some number theory. Notice what happens if we write all the divisors of 60 out as a product and then write them all out again backwards:

$$\begin{array}{rcccccccccccc} P &=& 1 &\cdot& 2 &\cdot& 3 &\cdots& 20 &\cdot& 30 &\cdot& 60 \\ P &=& 60 &\cdot& 30 &\cdot& 20 &\cdots& 3 &\cdot& 2 &\cdot& 1 \\ P^2 &=& (1\cdot 60) &\cdot& (2\cdot 30) &\cdot& (3\cdot 20) &\cdots& (20\cdot 3) &\cdot& (30\cdot 2) &\cdot& (60\cdot 1) \end{array}$$

  We can see now that the product of all divisors of an integer n is $n^{\tau(n)/2}$.
- Easier way to think of divisors of divisors problems. If we are asked find the product of the divisors of n that are multiples of m, we recognize that all divisors of n that are multiples of m and equivalent to md where $d = n/m$. Thus, we need only find the product of the divisors of d and then multiply in an m for each d.

Problems:

i. How many positive integers are positive divisors of exactly 2 of the 3 integers 840, 960, and 1200?

ii. Find the product of all of the positive divisors of 450 that are multiples of 3.

iii. Find the product of the divisors of 3200 that are perfect squares.

iv. Let m and n be two relatively prime positive integers. If m has 12 positive divisors and n has 10 positive divisors, what is the product of the positive divisors of mn?

v. What is the sum of all positive integers less than 100 that have exactly twelve divisors?

vi. Find the sum of the perfect square divisors of the smallest integer with exactly 6 perfect square divisors.

vii. Find the product of the positive divisors of 6480 that are multiples of 12.

Questions:

1. How can we use divisors to tell us about divisors? That is, in one problem, the solution observed that since some number n has 7 divisors, its prime factorization has one and only one prime. Can we restate the above information in a way that's more illuminating?

   I guess we can think about the chart. Since 7 is prime, we can write each divisor as the product of two numbers: 1 and some power of a prime. That is we have $p^0, p^1, p^2, p^3, p^4, p^5, p^6$ where I ommitted the 1 in each pair for convience.

2. Is their solution wrong? What if $p = 2$?

3. I solve the problem, "How many of the positive divisors of 45000 themselves have exactly 12 positive divisors?" with messy casework, but I imagine there's a way we could reword the problem to use combinations. How could we do that?

4.6 sPeCiAl NuMbErs

I was not a fan of this section. Here's a challenging problem:
https://artofproblemsolving.com/ebooks/intro-number-theory-ebook/par/117260

decent AMC problem:
https://artofproblemsolving.com/ebooks/intro-number-theory-ebook/par/117264

good AMC problem:
https://artofproblemsolving.com/ebooks/intro-number-theory-ebook/par/117272

Extra: Fibonacci stuffs

4.7 Algebra with Integers

Find the sum of all positive four-digit palindromes.

fun amc problem

4.8 Base Numbers

Fix in morning:
https://artofproblemsolving.com/ebooks/intro-number-theory-ebook/par/117680

Fix when you have wifi!

Don't forget!
Seriously, don't.

Ideas:

- Think of numbers as collections of bundles makes base problems easy. That is, we can think of 4321_9 as 4 bundles of 9^3, 3 bundles of 9^2, 2 bundles of 9^1 and 1 bundles of 9^0. Notationally, we write

$$4321_9 = 4 \cdot 9^3 + 3 \cdot 9^2 + 2 \cdot 9^1 + 1 \cdot 9^0.$$

- We can extend the idea to converting between bases. Notice, one bundle of 9^1 is equivalent to one bundle of $(3^2)^1$. We do, however, have to be careful about the quantity of bundles. In base three, we cannot have 6 bundles of 3^2 because

$$6 \cdot 3^2 = (3 \cdot 2 + 0) \cdot 3^2 = 2 \cdot 3^3 + 0 \cdot 3^2.$$

  In other words, $6_9 = 20_3$. We could use this idea to quickly convert between base 9 and base 3, but in practice, I think it's safest to write everything out. Doing so will ensure that we don't accidentally skip a power.

- We can use other bases to make base-counting[5] problems easier to solve. If we are asked to Find the nineteenth-smallest positive integer whose base-3 representation includes only the digits 0 and 1, we can use base 2 to simplify the problem. Notice, 19 in base 2 will be the 19th numeral in base 3 to use only 1's and 0's. So, all we need to do is convert 19 to base 2 and change the subscript to a 3.

- Multiplying a base numeral by what I will call the "fundamental base", that is, 10_b, is has the same effect as multiply a positive integer by 10. Why? It simply raises everything by a power, same as base 10. So,

$$123_b \cdot 10_b = 1230_b.$$

- Using our previous idea, we can see how many terminal zeros a given positive integer will have in a certain base by using prime factorization. For example, 7! has a prime

[5] Counting problems in which the base is not base 10. See problem i. for an example.

factorization $2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1$ which is equivalent to $14 \cdot 360$. Notice, 360 in base 14 is some numeral n. Then, we can write $7!_{10} = 10_{14} \cdot n_{14} = n0_{14}$ so 7! has one terminal zero in base 14. More generally, for some positive integer m, we will see that m will have k terminal zeros in base b if the prime factorization of m includes b^k. If this isn't clear, the base 10 equivalent says that if some positive integer m has 10^k in it's prime factorization, then clearly, m has 10^k terminal zeroes.

- I don't entirely understand this idea yet, but I call it "base converting trick shots". Essentially, if we recognize that a problem calls for conversion between bases in which one is a power of the other, we can use tricks to really speed up the process. See here.

Problems:

i. How many base-6 integers less than 1000_6 include 5 as a digit?

   A word of caution: This problem is harder than it appears; it should be solved with logic. Do *NOT* try to list out all the numbers. If you need to list out a few numbers to get a sense of what's happening, you should not use more than 36 numerals.

ii. How many positive integers require 3 digits when written in base 12, but require 4 digits when written in base 8?

iii. Find the one-hundredth-smallest positive integer that can be written using only the digits 1, 3, and 5 in base 7.

   I attmepted this problem. I have no idea how it works or how we find a solution without listing out numbers and using some counting along the way. I couldn't even make much sense of the solution.

   Edit: The solution makes sense, but it would be a nightmare to try this problem without first having done some counting. It's more of a counting problem than a number theory problem, in my opinion.

iv. How many of the 343 smallest positive integers written in base 7 use 4 or 5 (or both) as a digit?

   The author of this book does something that I claim are sort of similar to bijections between a base and some set of numbers that he's trying to count....and oh...wait, that actually sort of makes sense. It's still confusing, but I think I'm starting to get it. Anyway, we can solve this problem with simpler, ableit slower and more error prone counting techniques, and I find such ideas easier to prove so that's how I attempted to solve the problem. I made an error, but if I hadn't I would have gotten the same answer.

v. I can't even tell if I solved the problem they wanted me to solve or if I solved a different problem: bogus, v-man, it totally will!

vi. We have another weird bijection between different bases problem. Write it out on chalkboard when you get back.

Questions:

1. In problem 8.14 we see some algebraic methods to make quick conversions. Namely, we see that for $b \geq 3$ the following holds:

$$\begin{aligned} 100_{b+1} &= 1 \cdot (b+1)^2 \\ &= b^2 + 2b + 1 \\ &= 1 \cdot b^2 + 2 \cdot b^1 + 1 \cdot b^0 \\ &= 121_b. \end{aligned}$$

   I wonder why this works; can we generaize for converting between b and $b+n$? I imagine we would need to be careful about modular properties. As we increase the power in the binomial expansion the coefficients will increase and if they are greater than b then we will have a problem.

2. See problem iii.

3. See provlem iv. and bijections.

4.9 Base Number Arithmetic

Ideas:

- Arithmetic in base b arithmetic in base 10. However, our "fundamental base" is no longer 10 itself, but b. Observe:

$$\begin{array}{r} {}^{1\,1}\\ 623_8 \\ +\ \ 164_8 \\ \hline 1007_8 \end{array}$$

  Notice that 6+2 = 8 = 10_8 so we have to carry, just like we do in base 10 arithmetic.

  Here are some other examples:

$$\begin{array}{r} 31_9 \\ +\ 52_9 \\ \hline 83_9 \end{array} \qquad \begin{array}{r} 212_3 \\ +\ \ 102_3 \\ \hline 1021_3 \end{array} \qquad \begin{array}{r} 43_5 \\ +\ 124_5 \\ \hline 222_5 \end{array} \qquad \begin{array}{r} 4262_7 \\ +\ 1542_7 \\ \hline 6134_7 \end{array} \qquad \begin{array}{r} 1\,A\,7_{16} \\ +\ 2B\,7\,A_{16} \\ \hline 2D\,2\,1_{16} \end{array}$$

  Example and Code Credit: AoPS's *Introduction to Number Theory*

- Subtraction mostly works the same way:

$$\begin{array}{r} \overset{4}{\cancel{5}}\,\overset{11}{\cancel{1}}\,4_6 \\ -\ 4\,3\,3_6 \\ \hline 4\,1_6 \end{array}$$

  However, I'll add that when I do the computation, I'll probably use 7 instead of 11 beause $11_6 = 7_{10}$ which just makes the computation easier. I imagine there are some places where you need to be more careful because that might not work, but eh, for now, whatever :P

- Base number problem solving is easier if we break it into chunks. Consider the problem, Is there any base b for which b is prime? If so, provide an example. If not, explain why not."

  First, can we immediately qualify the nature of b given that 3 and 4 are represented in a base b numeral?

  Second, how could we use the definition of the prime number to help us with this problem?

  Third, can we use algebra and our previous point to solve the problem?

- Algebra can show up in unexpected places. How can we use difference of cubes to simplify the computation in the problem below?

  Find the largest prime number (in decimal form) that divides the sum:

$$1_2 + 10_2 + 100_2 + \cdots + 100000000_2.$$

- Some problems require that we find clever symmetry: bruh

Questions:

1. In the following problem, I simply multiplied the number by 3 in base 2, which gets the wrong answer. Why is my answer wrong?

   A binary number consists of 17 digits, all of which are 1. Triple the number.

2. Lol what is this?

   Very funny—and unusual—problem between Harris Pilton and Berris Fueller:

   The evil villain Harris Pilton wrote three secret two-digit numbers, x, y, z on a napkin. Berris Fueler must name three numbers, A, B, C after which Harris will announce the value of $Ax + By + Cz$. If Berris can then name Harris' three secret numbers, Harris will let Berris go. Save Berris! Come up with a way to choose A, B, C such that Berris is sure to escape Harris' evil clutches.

4.10 Units Digits

Problems:

i. How many of the positive divisors of 6^{2006} have a units digit of 6?

   This is a counting problem. Don't mess it up. Also, try to form your organized work before looking at how AoPS breaks the problem down. That is, don't open the link without trying the problem first.

ii. Prove or disprove the following conjecture:

   For some base b, and some perfect square s in base b, the only possible values of the units digit of the numerals that are a root of s sum to 10_b.

   Example: If the units digit of a perfect square in base 10 is 1, then the units digits of its root must be either 1 or 9 and $1 + 9 = 10$.

   Start with $b = 10$ and then generalize for arbitrary $b > 1$.

   Follow up question, can you characterize the cyclic nature of how units digits repeat for perfect powers?

   Clarification: Units digits of perfect powers of

   (a) 2: 4,8,6,2

   (b) 3: 3,9,7,1,3

   (c) 4: 6,4

   (d) 5: 5

   (e) 7: 9,3,1,7

iii. I have absolutely no idea how we are supposed to find that base 16 is the magic number system in which the universe swirls into the perfect shape we need to solve the following problem, but it (base 16) is. And boy does it deserve a video.

   Edit: I'm guessing base 16 is the first base such that we have nice powers of 4 and a nice units digit value.

   Determine all non-negative integral solutions $(n_1, n_2, \ldots, n_{14})$ if any, of the Diophantine equation

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1599.$$

4.11 Decimals and Fractions

I mostly skipped this chapter as the vast majority of it was review from pre-algebra. However, I suppose there were a few ideas/problems worth mentioning.

- In problem 11.8 we are told that gcd(a, b)=1 and we are asked to show that if b is divisible by a prime other than 2 or 5, then the fraction a/b is not a terminating decimal.

  The proof is...okay. We recall that if the decimal terminates, there is some power of 10 that we can multiply a/b by such that it is an integer. Since a, b are co-prime, if b has any prime divisor that is neither 2 nor 5, then the fraction clearly can't reduce, so it's not an integer and the decimal repeats. Wahoo.

- The book pointed out that we could write every decimal as an infinite geometric series. Not sure when I'd use that, but there it is.

- Radix is a term I should look up.

- **Problem 11.31** is worth looking at again:

  Suppose that a and b are digits, not both nine and not both zero, and the repeating decimal $0.\overline{ab}$ is expressed as a fraction in lowest terms. How many different denominators are possible?

- The extra section at the end of the chapter 11 challenge problems had a nice collection of interesting problems. Example:

  Prove that every odd square in the octonary system (base 8) has a units digit of 1. Prove also that if this units digit 1 is snipped off, the remaining part is always a trinagular number.

4.12 Intro to Modular Arithmetic

Ideas:

- There are several ways to test for modular congruence:

  1. I definitely messed this up somewhere in my notes, but here's a nice notation bit from AoPS:

     Modular arithmetic is a system for counting using only the integers

     $$0, 1, 2, \ldots, m - 1.$$

     When working in this system, we say that we are working with the integers modulo m. The number m is the modulus of the system.

  2. We can "directly" use the division algorithm—which is reliable way to minimize errors when working with negative values.
     Ex: Are -373 and 33 congruent mod 5? No!

     $$\frac{-373 - 33}{5} = \frac{-406}{5} \notin \mathbb{Z}.$$

3. We can "indirectly" use the division algorith. Let us check our previous work. Notice, we can write -373 as

$$\begin{aligned} -373 &= -375 + 2 \\ &= -63 \cdot 5 + 2 \\ &\equiv 2 \pmod 5 \end{aligned}$$

   Whereas 33 is clearly $3 \pmod 5$ so -373 and 33 are *not* congruent mod 5.

4. We can use the "clock method" but I find that that really only works for smaller numbers and for negative numbers, it's just likely to produce errors.

- Inequalities can help us speed up our work when we need to list out numbers. For example, if I want to list out the number of positive integers that are 1 greater than a multiple of 5 between 0 and 100, then I can simply write $0 < 5n + 1 < 100$ and solve for n to find $0 \leq n \leq 19$. Thus, there are 20 such integers (1, 6, $\cdots$, 91, 96).

- We say that r is the modulo-m residue of n when $n \equiv r \pmod m$ and $0 \leq r < m$.

- The integers congruent to some residue r make up the residue class. Do not confuse residue classes with residues. For some modulus m, there are m residues $(0, 1, 2, \cdots, m-2, m-1)$ whereas there are infinitely many integers in any given residue class.

- Occasionally it is nicer to work with negative integers rather than residues. Observe that $21 \equiv -1 \pmod{11}$ and $12 \equiv 1 \pmod{11}$ so $21^{100} - 12^{100} \equiv 1 - 1 \equiv 0 \pmod{11}$. Thus, we have determined in mere seconds that the obsurd expression $21^{100} - 12^{100}$ is a multiple of 11.

  A key idea here is to recognize when we have a number that can be written as a nice number when we choose to do so using a negative representation. In other words, if a number is one or two short of the modulus, that should suggest that the problem calls for using negative integers.

- Consider the following problem: find the units digit of 7^{7^7}.

  Though I recognize the AoPS is correct, I find it to be less illuminating than I would like. Let's see if we can fix that.

  Though this isn't necessary, I believe listing out some powers of 7 will make the solution more clear, so that's where we will start.

$$7^1 = 7 \quad 7^2 = 49 \quad 7^3 = 343 \quad 7^4 = 2401...$$

  We've actually already demonstrated a pattern, but sometimes if the ideas aren't clicking it helps to write out a few extra terms. However, as we continue to write out terms, the arithmetic will become more and more combersome so let's simplify. We really only care about the units digit, so we can ignore everything after it. We now have

$$7^5 = \_\_7 \quad 7^6 = \_\_9 \quad 7^7 = \_\_3 \quad 7^8 = \_\_1...$$

It should now be clear that a pattern has emerged. We can use the residue class of mod 4 to generalize the behavior of the units digit of all powers of 7. Namely, we see that the units digit of 7^n is 7 when $n \equiv 1 \pmod 4$, 9 when $n \equiv 2 \pmod 4$, 3 when $n \equiv 3 \pmod 4$, and 1 when $n \equiv 0 \pmod 4$. We have now reduced the complexity of our problem as we can now determine the units digit for any 7^k given that we know the residue that corresponds to k in mod 4. In this case, k is 7^7. In mod 4, $7 \equiv -1$ so we have $7^7 \equiv (-1)^7 \equiv -1 \equiv 3 \pmod 4$. Thus, we conclude our problem by noting that 7^{7^7} has the same units digit as 7^3, which we know is 343.

Note: Our solution is much easier to follow as we've written it above, but if we use AoPS's method, we can clean up the problem quite nicely. Borrowing from their solution, we write

$$\begin{array}{lcl}
n \equiv 0 \pmod 4 & \Rightarrow & 7^n \equiv 1 \pmod{10} \\
n \equiv 1 \pmod 4 & \Rightarrow & 7^n \equiv 7 \pmod{10} \\
n \equiv 2 \pmod 4 & \Rightarrow & 7^n \equiv 9 \pmod{10} \\
n \equiv 3 \pmod 4 & \Rightarrow & 7^n \equiv 3 \pmod{10}
\end{array}$$

so, using our earlier work we can say

$$7^{7^7} \equiv 7^3 \equiv 3 \pmod{10}$$

which means 7^{7^7} has 3 as its units digit as desired.

Problems:

i. I find it unlikely I would have gotten to the solution using modular methods if I did not have the guard rails of knowing that this problem appears in a chapter on modular arthimetic. However, the problem demonstrates the power of such methods, so into the collection it goes.

ii. Find the tens and units digit of 7^{2006}.

iii. Find the smallest positive integer n such that

$$617n \equiv 943n \pmod{18}.$$

iv. Ryun and Zhenya play a game with a pile of 82 toothpicks. The players take turns removing 1, 2, 3, or 4 toothpicks from the pile on each turn. The player that removes the last toothpick loses. Zhenya goes first. Help her formulate a winning strategy.

v. Much like the game between Ryun and Zhenya, this is more of a logic puzzle than a math puzzle but okay A group of 25 Chicago Cubs fans got together for a party to discuss how this is the year. At the end of the party, many of the Cubs fans shake hands with one another. Let n be the number of the Cubs fans who shook hands with an odd number of other Cubs fans. Prove that n is even.

vi. Find the remainder when

$$10^{10} + 10^{100} + 10^{1000} + \cdots + 10^{10000000000}$$

is divided by 7.

vii. What is the size of the largest subset, S, of $\{1, 2, 3, \ldots, 50\}$ such that no pair of distinct elements of S has a sum divisible by 7?

viii. Find all prime numbers p for which $p^2 - 1$ is not a multiple of 24.

Questions:

1. Find the remainder when
$$1^2 + 2^2 + 3^2 + \cdots + 99^2$$
is divided by 9.

   I solved correctly first time, but I did not use negative numbers. They suggest an alternative method using sum of squares, but I got the wrong answer using that method. Therefore, I am confusion.

2. Modular stuffs I simply multiplied by 15 so I got $15n \equiv 15 \pmod{60}$ so $n \equiv 1 \pmod{60}$. Is that valid?

4.13 Divisibility Rules

Ideas:

- Notice that an integer is congruent modulo 10^m of its last m digits. Then, since 2^m is a divisor of 10^m we must have that some integer is congruent modulo 2^m to its last m digits. Observe.

  Let n be some integer such that $n = k \cdot 10^m + n_m \cdot 10^{m-1} + n_{m-1} \cdot 10^{m-2} \ldots n_2 \cdot 10^1 + n_1 \cdot 10^0$. Or, if we wish to write in a way that is less rigorous but easier to read, let

  $$n = k \cdot 10^m + n_m n_{m-1} \ldots n_2 n_1$$

  where $n_m n_{m-1} \ldots n_2 n_1$ represents an integer such that the ith digit of the integer is the ith digit of n.

- We can prove divisibility tests by rewriting integers using modulo residues. 10 is congruent to 1 mod 3 and mod 9. Thus, the classic "sum the digits" test can be proved by rewriting the powers of 10 for each digit in mod 3 or 9. Then, it's clear that if the sum of the digits is congruent to 0 mod 9, then the number is divisible by 9. In fact, we can do better. The remainder of a number n divided by 9 is the same is the residue of the sum of n's digits mod 9.

  We can do something similar for other numbers like 11. We see that $10 \equiv (-1) \pmod{11}$ so an integer is divisible by 11 if the alternating sum of its digits is divisble by 11. Ex: 1958 is divisible by 11 because 9 + 8 - 5 - 1 = 9 + 2 = 11. Similarly, 40601 is divisible by 11 because 0 - 4 - 6 - 1 = -11.

- Sometime LCM is more powerful than divisibility tests, as seen here: Find a six-digit number whose first three digits are 523 such that the integer is divisible by each of 7, 8, and 9.

  I suppose when working with problems, we can mentally approximate the difficulty difference between the two and choose the more optimal path.

Questions:

1. Can we not alternate as we wish with the 11 divisbility test? Meaning, does it really matter if we start with positive or negative? Given some alternating sum, if we multiply the whole sum by -1, all we do is flip the end result.
2. In this problem they prove a biconditional without directly proving both sides. Or at least, that's how it appears. I suppose it's possible their work implies both, but I would like to know more. I believe this is the 7 and 31 divisibility test problem.
3. Supposedly we use divisibility rules to do some stuff with repeating decimals, but I don't really see it: Show that for $n \geq 2$, the number of digits in the block of repeating digits in the decimal expansion of $\frac{1}{3^n}$ is 3^{n-2}.

Problems:

i. Prove that a power of 2 cannot end in four equal digits.

4.14 Linear Congruences

Ideas:

- An integer n **has an inverse** in modulo m if and only if m and n are co-prime.

Need to write proof! See here to get started. Also, notice that we are playing with ideas quite similar to those in chapter 4 of S's Algebra text (Euclid's Theorem).

Proof. (Inspired by material from AoPS)

($\Rightarrow$) If r has an inverse modulo-m, then there exists some integer s such that

$$rs \equiv 1 \pmod{m}.$$

If such a congruence is possible, we can write it as $rs = tm+1$ which implies $rs-tm = 1$. Then, let $d = \gcd(m, r)$. Since $d|r$ and $d|m$, we must have $d|1$, but that implies $d = 1$. So, if r has an inverse modulo-m, then $\gcd(m, r) = 1$.

This proof isn't really correct! It is, but it isn't. The second argument is effectively a group theoretic argument but it doesn't use group theory (because I didn't know group theory well enough yet). Use the cyclic group to rigorously write what you wrote.

($\Leftarrow$) Let $r < m$ and let m, r be co-prime. Because r, m are relatively prime, by definition, there is not integer n such that $rn = m$. If we observe the integers modulo m, then, for $x \not\equiv y \pmod{m}$ we must have $xr \not\equiv yr \pmod{m}$. Consider, then, what happens when we multiply r by each modulo-m residue. From our earlier work, we can see that each product will be different, so one of them must be one. Equally important, only one such product can be congruent to 1. Thus, if m, r are co-prime, r has a single inverse modulo-m.

We have now shown both directions, so the proof is complete. □

- One of the most important conclusions we can make from the previous idea is that **we don't always need to find an inverse to use it**. If we know an integer is relatively prime to the modulus and we wish to solve a congruence, we need only rewrite the current congruence such that the ingeter we need to remove appears on both sides. For instance, if our congruence is

  $$7x \equiv 9 \pmod{12},$$

  then we simply notice that

  $$7x \equiv 9 \equiv 21 \pmod{12}.$$

  Then, we multiply both sides by 7^{-1} and get

  $$x \equiv 21 \cdot 7^{-1} \equiv 3 \cdot 7 \cdot 7^{-1} \equiv 3 \pmod{12}.$$

- **Finding the inverse** modulo-m for a given integer is a task best left to computers—which is Travis speak for, "This took way longer than I thought it would and I'm mad at my inability to do something that isn't even hard." However, it's still important that we know how to do it.

  Let's say we want to find the inverse of 7 modulo-32. We could add 7 to 1 until we get to a multiple of 32, but that might take a while. If we're lucky, we will be able to recognize how to manipulate 7 into a number that is congruent to -1 mod 32. In this case, such a manipulation is $9 \cdot 7 = 63 \equiv -1 \pmod{32}$. We can then multiply both sides by negative 1 to get $7 \cdot -9 \equiv 1 \pmod{32}$. We see that -9 mod 32 is congruent to 23 so 23 is the inverse. We can check using what I guess is called the extended Euclidean Algorithm WHICH IS NOT EXPLAINED VERY WELL...sometimes. Othertimes, it's fine. (I struggled more than I would like). Such a process goes as follows.

  We first write the same numbers in a bunch of different forms, because it's completely clear that that's useful.

$$32 = 4 \cdot \underline{7} + \underline{4} \tag{3}$$
$$7 = \underline{4} + \underline{3} \tag{4}$$
$$4 = 3 + 1 \tag{5}$$

  Next, we take those same numbers and then write them again, but in different froms from last time...again, because it's not confusing at all and obviously useful.

$$\begin{aligned} 1 &= 4 - 3 \\ (4) \Rightarrow 1 &= 4 - (7 - 4) \\ 1 &= 2 \cdot 4 - 7 \\ (3) \Rightarrow 1 &= 2 \cdot (32 - 4 \cdot 7) - 7 \\ 1 &= 2 \cdot 32 - 9 \cdot 7 \end{aligned}$$

  We can now write that final equation in mod 32 to get

$$0 + (-9) \cdot 7 \equiv 1 \pmod{32}.$$

  Though it was sort of messy, we got the same answer as above. Whippee. Okay, never want to type that stuff out again. Also, to future readers, I was...quite hangry today so I hope that explains my rather unsual grumpiness. Let's move on.

- The congruence $ax \equiv b \pmod{m}$ has **no solutions** when $\gcd(a, m) \nmid b$.

  I believe this is pretty clear when written with actual numbers, but we'll do a generic proof anyway. Since $ax \equiv b \pmod{m}$ we can write $ax - mn = b$. Then, let $d = \gcd(m, n)$. If we write

$$\frac{ax}{d} - \frac{mn}{d} = \frac{b}{d}$$

it is clear that the left hand side is an integer and the righthand side is an integer if and only if $d|b$. So, if d does not divide b, the congruence has no solutions. $\square$

- Though there is **no division** in modular arithmetic, there are two methods that are similar. In particular, if the modulus and the congruence both share a common factor, we can take it out (first method):

$$12x \equiv 36 \pmod{44} \Rightarrow 3x \equiv 9 \pmod{11}.$$

Next, if we write things in the proper form—and luckily that's already done for us here—we can apply our second method, multiplying both sides by an inverse:

$$3x \equiv 9 \pmod{11} \Rightarrow x \equiv 3 \pmod{11}.$$

- In the book, there is a General vs. Applied Process outline that summarizes just about everything that I wrote above in a neat and comprehensive manner. The outline is some of the best writing in the book so far. One of the most important points will save you a lot of trouble if you remember to apply it correctly:

  A linear congruence $ax \equiv b \pmod{m}$ has solutions if and only if $\gcd(a, m) \mid b$.

- Symmetry is our friend. Consider the system of linear congruences:

$$\begin{aligned} N &\equiv 2 \pmod{4} \\ N &\equiv 2 \pmod{7}. \end{aligned}$$

We see that $N - 2$ is a multiple of both 4 and 7, so it must be a multiple of 28. Therefore, $N \equiv 2 \pmod{28}$.

Questions:

1. I've seen different concepts that are both called the Euclidean Algorithm. It may be the case that they are effectively doing the same thing and I'm simply failing to realize how. All the same, it's frustrating. Namely, we have what AoPS denotes as the Euclidean Algorithm $\gcd(m, n) = \gcd(m - n, n)$ and we have the "divide, take the remainder, then do the same thing over and over again" approach. How are the two related?

2. Why no 53? Why 40? Bruh. Can we not factor out like we did earlier?

   I think because we can't factor n.

4.15 Number Sense

Ideas:

- Problems that require integer solutions but have fractions in an expression should immediately ask us if we can restrict the possibilities somehow. See if you can use that to solve this problem: Find all ordered triples of positive integers (a, b, c) such that

$$a + \frac{1}{b + \frac{1}{c}} = 9.5.$$

- Though it can be tempting to use powers of integers, sometimes it's easier to represent integers as the product of some other (nicer) integer and a fraction. If we wanted to calculate $163 \cdot 125$ in our head, we could rewrite the product as

$$163 \cdot 1000 \cdot 1/8 = 163/8 \cdot 1000 = (20 + 3/8) \cdot 1000 = 20.375 \cdot 1000 = 20375.$$

  We can also represent integers as sums. Notice, we can rewrite $2350 \cdot 9$ as

$$2350 \cdot (10 - 1) = 23500 - 2350 = 21150.$$

  Don't stress about this technique too much. Occasionally it can come in handy, but for now, I find it to be more of a flashy slam dunk than fade away—fun, but not as useful.

- We can expand our previous idea by utilizing the **difference of squares**. If we can write a number as the difference of two squares, we can express the number in a form that's likely easier to manipulate.

  Recognizing binomial coefficients is also quite helpful.

- Don't forget to utilize divisibility tests of higher primes like 7,11, and 31!

  Might want to go back and prove some of those :P

- Symmetry! If we have a problem where all terms of interest can be written with the same residue, we likely have a problem that involves the LCM as a solution.

- Representing numbers in different forms to make problems easier is no new idea by now. In addition to products, fractions, sums, and decimals, we can represent numbers in modular form (or even base form). Don't forget it!

Problems:

i. What is the least integer, greater than 1, that is a factor of $11000 + 1100 + 11$?

ii. How many digits are in the product $8^{12} \cdot 25^8$? Don't forget all your methods. Sometimes a problem requires more than one method!

iii. I don't want to delete all the RGB code so here's the problem link as is...

iv. Find the least positive integer n for which $\frac{n-13}{5n+6}$ is a non-zero reducible fraction.

v. The numbers from 1 to 2002 are listed in the following order: First all numbers that are not divisible by 3 are listed in (increasing) order. Then all numbers that are divisible by 3 but not by 3^2 are listed in order. Then all numbers that are divisible by 3^2 but not by 3^3 are listed in order, etc. What is the last number in the list? (Give the entire number, not just its last digit.)

Questions:

1. Where did the 1000 go?...Did...did they skip 1000 iterative steps in a row as if it were trivial? I mean, once you see it, I guess it is, but I mean damn.

Need to come back to this

Problem solving Strat: when stuck, reword the problem using what you already know to form a new sub problem that's easier to solve

https://artofproblemsolving.com/ebooks/intro-number-theory-ebook/par/120233

Lost pen :(

but it was just about empty so I guess that's fine

Finished March 19

5 Intro to Geometry

Started March 20

5.1 What's in a Name?

Ideas:

- Don't confuse **lines** and line **segments**. A line is infinite, a segment, finite.

- A **chord** is simply a segment in a circle (a line segment that connects two points on a circle). Do not confuse a chord with a secant line. A **secant line** is like a chord, but it's a line, so it's infinite (whereas a chord is finite).

- Segments (and lines) can intersect a circle in at most two points.

- We **construct** geometric figures using only a straightedge (not a ruler!) and a compass. The ability to construct is a central skill in understanding geometry. While constructing we may do the following:

  1. Given a point, you can draw any line through the point.
  2. Given two points, you can draw the line that passes through them both.
  3. Given a point, you can draw any circle centered at that point.
  4. Given a point and a segment, you can draw the circle with its center at that point and with radius equal in length to the length of the segment.
  5. Given two points, you can draw the circle through one point such that the other point is the center of the circle.

Problems:

i. Two circles and three straight lines lie in the same plane. If neither the circles nor the lines are coincident (meaning the two circles are different and the three lines are all different lines), what is the maximum possible number of points at which at least two of the five figures intersect? Don't confuse "exactly" and "at least"!

5.2 Angles

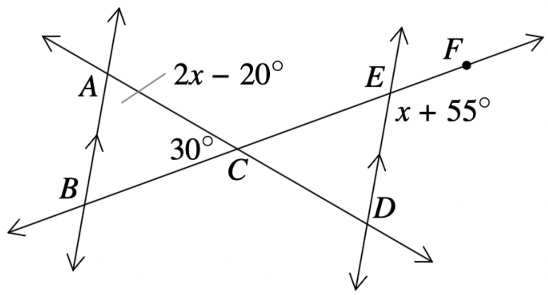
Ideas:

- Supplementary $\Rightarrow \angle A + \angle B = 180$ whereas complementary $\Rightarrow \angle A + \angle B = 90$.

- If we already have two lines that are parallel, drawing another line parallel to them both can sometimes quickly lead us to a solution (when our line is placed in a clever way).

- We don't need rectangles to prove that the angles of a triangle sum to 180. We need only a fourth line parallel to the base at the top vertex of the triangle. From there, we can use alternate interior angles.

- Exterior angles can rapidly speed up a problem solving approach.



Notice that if we draw a point G right below B on $\overleftrightarrow{AB}$ then $\angle GBC$ is congruent to $E = x + 55$. Moreover, $\angle GBC$ is an exterior angle so

$$\angle GBC = 2x - 20 + 30 \Rightarrow x + 55 = 2x + 10\,\text{deg} \Rightarrow x = 45.$$

- **Corresponding Angles Theorem:** Draw three lines such that one line intersects the two others. Let there be two angles x and y such that x and y have the same relative position at each intersection (if x in top right, then y in top right on other intersection). We will show that if $x = y$, then the "two other" lines must be parallel.

  Suppose that the two other lines are not parallel and that $x = y$. Then, we may form a triangle with two angles that sum to 180, so we have reached a contradiction (some day I should come back and draw this so it's clearer, but you get the idea). Thus, if $x = y$, then the two other lines are parallel, as desired.

- The summary section defines many different types of angles. If you need to remember names, that's a good place to start.

Problems:

i. Prove the Alternate Interior Angles Theorem

ii. Should probably do this one. It's a clock angles between the hands problem.

iii. One angle of a triangle is equal to the sum of the other two. Show that the sum of two exterior angles of the triangle is 180° greater than the third.

iv. Self composed problem:

   Use our visual proof of the sum of interior angles of a convex polygon as inpsiration for an inductive proof that the sum of the interior angles of any of any convex polygon with n sides is $180(n-2)$.

v. Prove that the sum of exterior angles of any convex polygon is 360°. Do not use a circle; use what we know about exterior angles in triangles.

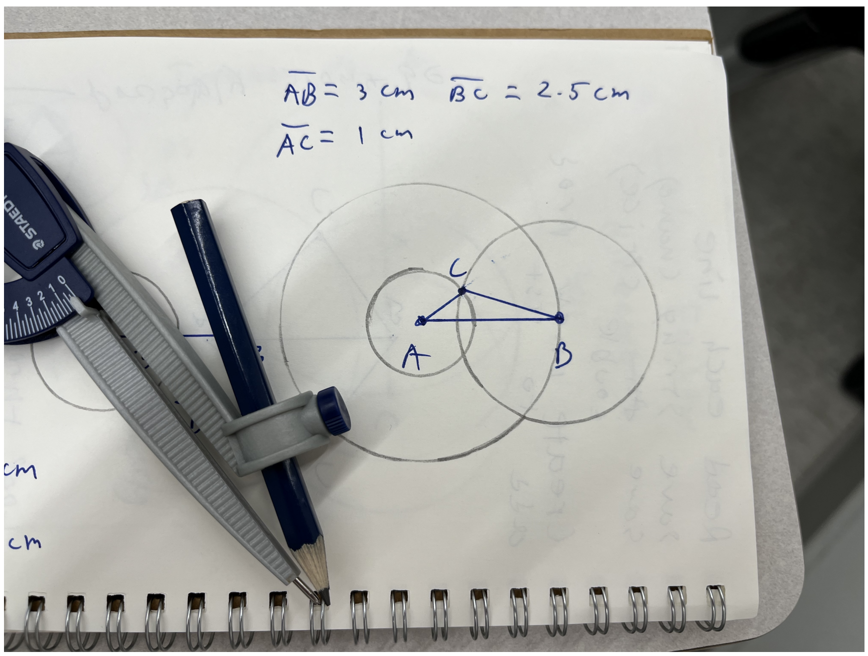
vi. Difficult angle problem

   You absolutely need to revist this problem. I don't know why this problem was so difficult for you, but it shouldn't have been.

5.3 Congruent Triangles

Ideas:

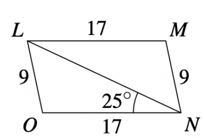
- **Constructing triangles:** Let's say we have a triangle $\triangle ABC$ and we want $\overline{AB}$ to be 3 cm, BC to be 2.5 cm, and AC to be 1 cm. Then, we can draw a base (I don't think it matters but I picked 3 cm) and then we draw all the points that could be C relative to A and all the points that could be C relative to B. Then, the intersections form our triangle. Like so:



- **SSS congruence** tells us that if two triangles have the exact same side lengths, then they are congruent. Our construction exercise has not rigorously proved this, but it has given us the intuition.

- **CPCTC:** Corresponding parts of a congruent triangle are congruent. We can use CPCTC in a variety of problem solving methods.

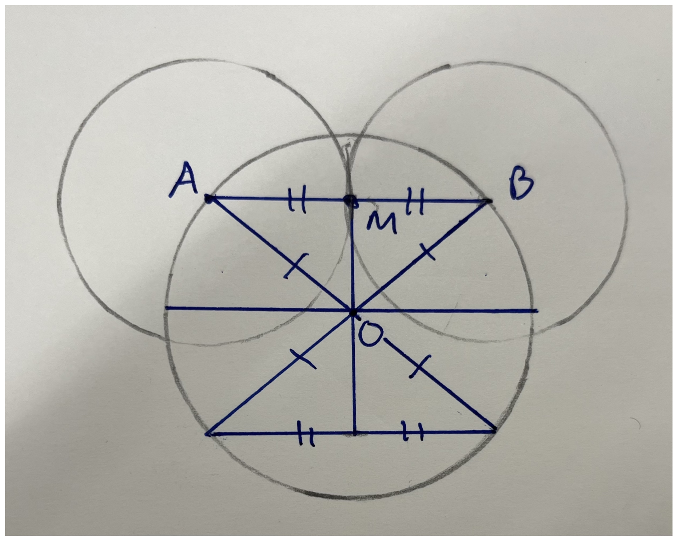
- We can sometimes use SSS congruence to determine if lines are parallel. Consider a parallelogram that has not already been specified as a parallelogram. If we are given the lengths of the sides, we can say that the two triangles formed by drawing a diagonal are congruent, as are the alternate interior angles (via CPCTC), which tells us the base and the...top? are parallel. See below for a visual (credit to AoPS).



- Using SSS congruence and CPCTC we can prove that **if a radius bisects a chord that is not the diameter, then it is perpendicular to the chord**. Notice in the diagram below, the sides give us SSS which allows us to use CPCTC to say

$$\angle AMO = \angle BMO \Rightarrow \angle AMO = 90°$$

because $\angle AMO + \angle BMO = 180°$.



- We can use SSS to get some intuition about **SAS congruence**. Side-angle-side tells us that if two sides and the angle between them are equal to the corresponding sides and angle of another triangle, then the two triangles are congruent. If you construct a base, a circle on one end of the base, and draw two rays from the center of the circle to two points such that the angle is the same for each, what are the final steps to recognize the intuition of SAS?

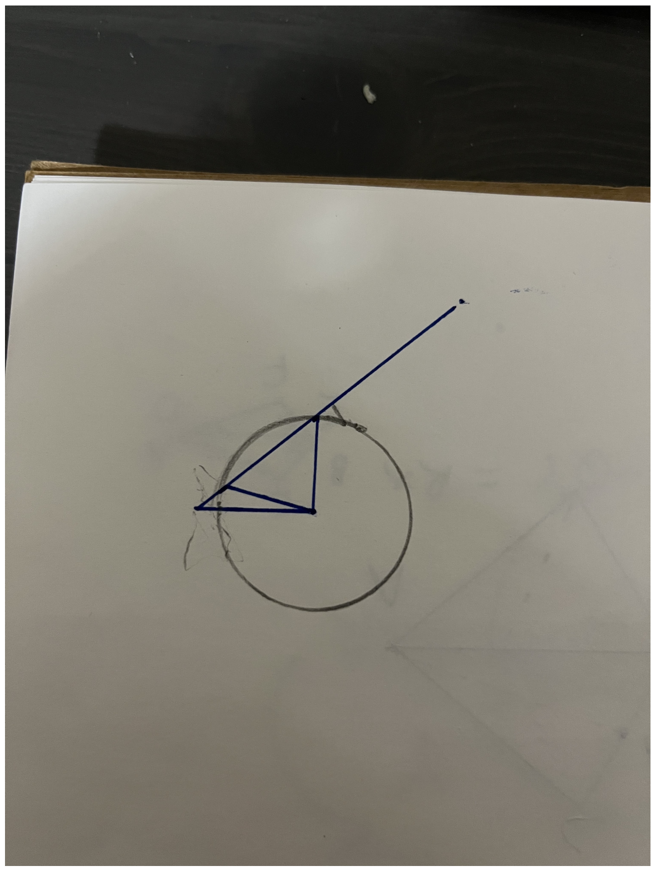
- **ASA congruence** is, in my opinion, rather obvious. If two angles of one triangle and the side between them are equal to the corresponding angles and side of another triangle, then the two triangles are congruent. I imagine this is the first congruence relation that we could immediately prove. Though I will leave it for later, I believe we could use contradiction. Suppose that ASA doesn't imply congruence. Can we compose an impossible triangle?

- **AAS congruence** tells us that if two angles and a side of one triangle equal the corresponding angles and side in another triangle as shown below, then the triangles are congruent.

  Think about it this way. If we have two angles, we have the third. Then, if we have a length, what does that imply about the triangle's structure? How could we compose another triangle with the same properties that isn't congruent? (We can't).

  There is a **warning** to be aware of, though. Notice that corresponding has been showing up a lot. If two triangles have two equal angles and one equal side but the side is of a different orientation, then we cannot use AAS to prove congruence. See here for a visual.

- SSA is *NOT* a valid congruence. Draw a base. Then construct a circle with a radius that is slightly less than the base. Now, on the other side of the base, draw a ray such that it intersects twice with the circle. From there we can see that there are two triangles with SSA but the two triangles are clearly quite different. See below:



- Equilateral triangles are isosceles triangles in disguise. Say we have a triangle with three equal sides. We can prove that the three angles are equal by picking a point, and noting that the sides obtained from that point are equal, so their corresponding angles are equal (isosceles). Then we can do the same thing again and prove the third angle is equal to the other two.

- **Construction is a two part process.** Once we spot how to make the construction, we must then prove that the construction is valid. Proving is typically the easy part. So, play around. Draw out the full circle. See where nice intersections occur. Make a conjecture. Try to prove it. Repeat if necessary.

Problems:

i. In the figure below, $AD = CD$ and $AB = BC$. Prove that $\overline{DB} \perp \overline{AC}$.

ii. You need to actually draw it out. Using their diagram isn't enough.

iii. Squares, equilaterals, and isosceles.

iv. Chaining galore! The problem linked here is both considerably more and less difficult than it seems...somehow. It requires great organization, a lot of space, decent tools (compass and colored utensils), and a good understanding of how we can combine different techniques. With all that—and some patience—it should be quite doable.

v. I made this harder than it needs to be; I was staring right at the answer but I got so lost in trying to find congruences that I unknowingly forgot what we were looking for. Remember, you can always right down what you've discovered, and come back after looking at another problem.

Questions:

1. How can I get better at thinking about geometry without a diagram? Most of the solutions are solved symbollically and I find that extremely difficult to follow as I need to keep track of significantly more...objects? Anyway, I don't understand the proof of the AAS theorem as much as I would like to. I need to revisit it, and help would probably be good.

2. Can we use ASA and form a relationship between XR and QY? Can we not just do a sort of algebraic sum? Since they are the same? and the angles that define them are the same, wouldn't a change in one imply a change in the other?

3. We are asked to prove two equal angles on the same base implies an isoceles triangle. I drew a line that bisected the top angle so we can use AAS. However, I'm not sure that works because the logic makes an assumption that, though it might be correct, we haven't proved it yet. Need to check. Mmmm checking my diagram, I think it might work. Need to ask a professor. All the same, choosing to draw a line such that we get 90 degrees is better. I didn't know we were allowed to do that yet.

4. How can we develop an intuition for when something is or is not valid when it isn't incredibly obvious just by looking at the picture? (I would argue that that doesn't count anyway as that requires approximation, not logic.)

   Examples: Proving SSA false, we used circle and two points. Proving Isosceles we could decide to bisect angle or draw 90 degrees.

   See solution for this problem as another example

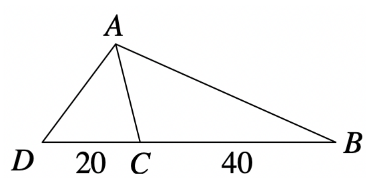
5. Can we use bisection twice on imaginary triangles to form what we want?

5.4 Perimeter and Area

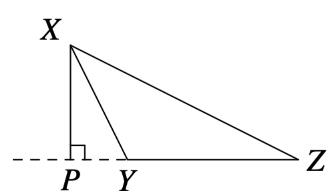
Ideas:

- Comparing area with **ratios** is a very powerful approach.
- A common theme in geometry is spotting the invisible, the implicit. One example of this is noticing that when triangles have either the same base or the same height, we don't really need it to learn a thing or two about their area. Notice in the diagram below, we can find the ratio between the areas of two triangles in the diagram without knowing their area or their height by observing that their height will cancel in the ratio:

$$\frac{[ABC]}{[ABD]} = \frac{(BC)(h)/2}{(BD)(h)/2} = \frac{BC}{BD} = \frac{2}{3}.$$



- As discussed later, we don't always need to form a perpendicular inside the triangle to calculate the area. Consider the triangle below:

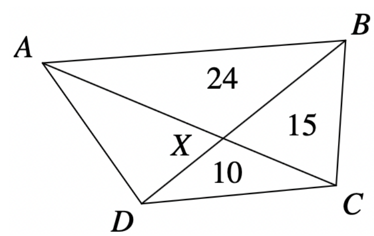


Clearly, the height is outside the triangle. Does $\frac{bh}{2}$ still work? Yes. We can prove it so using subtraction:

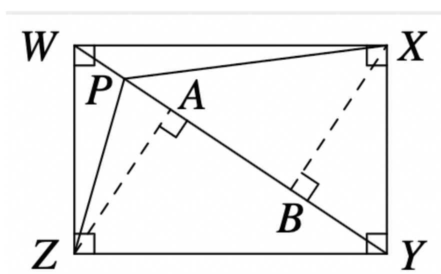
$$\begin{aligned}
[XYZ] &= [XPZ] - [XPY] \\
&= \frac{(XP)(PZ)}{2} - \frac{(XP)(PY)}{2} \\
&= \frac{XP(PZ - PY)}{2} \\
&= \frac{(XP)(YZ)}{2}.
\end{aligned}$$

To remember the proof, imagine we draw a rectangle $XPZQ$ around $\triangle XYZ$. Then, the area of $\triangle XYZ$, denoted $[XYZ]$ is the area of $XPZQ - [XPY] - [XZQ]$. Notice $\triangle XPY$ and $\triangle XZQ$ are both right triangles, so their area is quite simple to understand and prove.

- You should remove the notion of using a perpendicular bisector *inside the triangle* as the height of the triangle. It isn't necessary, and it it blinds your vision. Notice, in the diagram below, it would be quite difficult—if not impossible—to prove that $\triangle ABX$ and $\triangle BXC$ share a height using perpendicular bisectors inside each triangle. If we look at the problem differently, it's already implicitly in the diagram for us. Namely, both triangles share $\overline{BX}$; using that as the place to find the height, we can find the area of $\triangle AXD$ using ratios.



- In some same base/height problems we need to know about the other component (base/height). Occasionally it can be difficult to visualize. Fixing the difficult can often be as simple as extending lines. For instance, if two triangles share the same base, to visualize their height we extend their base and draw the height down from the vertex opposite of the base (seen in the diagram below).



5.5 Similar Triangles

Ideas:

- We can use **AA similarity** to demonstrate that two triangles are similar. That is, if two angles of one triangle equal two angles of another, then the triangles are similar.

- We can use **SAS similarity** to determine similarity. If two sides of a triangle are in the same ratio as two sides of another triangle and the angles between the sides are the same, then the two triangles are similar.
- We don't see it often, but we can use **SSS similarity** as needed.
- If two similar triangles have sides that differ by a factor of k, then the bigger triangle is k^2 times larger than the smaller triangle.
- When you're stuck on a problem, ask yourself, 'What piece of information have I not used?'

Problems: Lol all of them

Questions:

1. Similarities give us ratios. If our ratios, tell us $v_1 = v_2$ and $v_2 = v_3$, then by transitivity, shouldn't $v_1 = v_3$? Why did they do all that extra work?

5.6 Right Triangles

Ideas:

- If two sides of a right triangle are proportional to corresponding sides on another right triangle, the triangles are similar. I find this trivial, but we should remember it.

  Edit: It's not trivial :P

  How can we use HL or LL congruence to prove that if a radius is perpendicular to a chord, the radius bisects the chord? (and vice versa)
- Pythagorean triples can be generated. The general form is a pain. Some, but not all, can be generated using the following table:

n	$2n$	n^2-1	n^2+1
2	4	3	5
3			
4			
5			
6			

- **Heron's Formula**, which I need to rederive until I can do it without help, can be used to find the area of a triangle when we have all three sides but not the height:

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}.$$

- What do we use to construct perpendicular lines? Circles!
- Geometric eye = perpendicular lines and irrational numbers.
- Problem to look at: https://artofproblemsolving.com/ebooks/intro-geometry-ebook/par/2079

5.7 Special Parts of a Triangle

Ideas:

- The line segment from a vertex of a triangle to a point on the opposite side is called a **cevian**.
- The **perpendicular bisector** of a segment is the straight line consisting of all points that are equidistant from the endpoints of the segment.

  I do not have a completely flushed out understanding yet, but here's how I think about the perpendicular bisector and the necessity of equidistant segments.

  Consider two points A and B and the segment that joins them. Let M be the midpoint of AB. Let us draw some segment CM and consider what happens when we move C around. Suppose C starts such that it is equidistance from both A and B. If C moves closer to one or the other, what happens? WLOG let C move closer to B. Then, something must have changed in our setup. Since it is no longer equidistance, we no longer have congruent triangles...etc. At least, I think. I mean, I'm confident that's right, but I need to prove it, and I'm not going to do it now.

  Edit: I don't want to read whatever it is I said because I'm a lazy bastard, BUT I can provide a concise explanation of the whole equidistant thing. Since a perpendicular bisector divides a segment in half *and* is perpendicular to the segment, we can always form congruent triangles between the segment endpoints and any point on the perp bisec via SAS congruence; first side is half segment, angle is 90 degrees, second side is side the two triangles share (perp bisector).
- Consider some triangle $\triangle XYZ$. Then, the line that contains all points that are equidestiant from the rays $\overrightarrow{XY}$ and $\overrightarrow{XZ}$ is the **angle bisector** of angle X.
- We call three or more lines that meet at a single point **concurrent**.
- The **circumradius** of a right triangle is half the hypotenuse.
- Three (or more) points are **collinear** when we can draw a single line through all of them.
- Three noncollinear points form a circle.

- The angle bisectors of a triangle are concurrent at the triangle's **incenter**. The incenter is equidistant from the sides of the triangle. We call said distance the **inradius**. Naturally, we can inscribe a circle inside the triangle using the inradius. We call such a circle an **incircle**.
- The angle bisector theorem enables us to solve problems that concern both lengths and angle bisectors.
- Remember s (semiperimeter) from Heron's Formula? It's back. The area of a triangle can be calculated by multiplying its inradius by its semiperimeter. That is, for some $\triangle XYZ$ with an inradius r, we have

$$\begin{aligned}[XYZ] &= \frac{r(XY)}{2} + \frac{r(YZ)}{2} + \frac{r(XZ)}{2} \\ &= r\left(\frac{XY + YZ + XZ}{2}\right) \\ &= rs.\end{aligned}$$

  Note: The inradius is perpendicular to all sides. See a diagram here.

Problems:

i. Lol I put everything in my Putnam study guide at this point

Questions:

1. "The center of any circle that passes through all three points must be on all three perpendicular bisectors of the sides of the triangle with these points as vertices. Therefore, the circumcenter is the only possible center of a circle through the three points. Therefore, the circumcircle is the only circle that passes through all three points."

   How difficult is this to (rigorously) prove?

   I think we could do a proof by contradiction. We suppose that there could be a different circle, which implies different (noncongruent) triangle, which doesn't make sense because if we use the same three points, how could we get a triangle that isn't congruent?

2. This may be an analytical geometry question, but could we give a more intuitive explanation of equidistant properties and bisectors using algebra? Namely, we can define two angles that sum up to 180 as 90 - θ and $90 + \theta$, can we not? Then, how can use that to relate everything else on the triangle?

3. Can we use this to help us solve the olympiad diagnostic problem? Also, what does it tell us about the utility of angle bisectors and the angle bisector theorem.

4. The **median** is a segment drawn from the vertex to the opposite side of the triangle such that it divides the side in two (from a vertex to a midpoint). The point at which all medians are concurrent is called the **centriod**.

5. When given the lengths of the sides of a triangle in a problem, always take the time to check if it is a right triangle. Special properties of right triangles often simplify problems.

6. Medians create triangles with equal bases; therefore, they create triangles with equal areas. We can use these equal areas to learn even more about the medians of a triangle.

7. Altitudes are concurrent at the **orthocenter**.

Yeaaaah this is a mess but I don't know what to do about it...oof

6 Intermediate Algebra

6.1 Basic Techniques for Solving Equations

Ideas:

- Solving equations or systems of equations mostly comes down to elimination, isolation, substitution, or a combination of the three. Sometimes it may not be clear how we can do any of the three (or of their combinations) so we may need to apply clever tricks.
- Clever tricks take many forms, but here are a few. To eliminate/simplify, we can sometimes let our variable be equalt to 0 or 1 (or other nice values relative to the problem). We must be careful, though, as sometimes this creates an equation with no solutions. Always double check. We are free to do anything legal such as adding and subtracting the same number or multiplying by a scalar. Both can be used as prep work for various simplification techniques. We may also add/subtract the same number to both sides; one particularly sneaky way of doing this in systems of equations is to use our current equations to make a new equation, and then subtract/add said equation to one of the equations in our system. We can do this because we are effectively just adding the same number to both sides, even if symbolically it appears otherwise. Another helpful trick is exploiting symmetry (see intro Algebra).

Problems:

i. Suppose that a, b, c, d, e, f are constants such that $ax^2 + bx + c = dx^2 + ex + f$ for all values of x. Prove that $a = d$, $b = e$, $c = f$.

Questions:

1. I don't understand their question about reflection.
2. They referred to trig and triangles in this very symmetric problem. I don't know what my question is, but I know there is one.

6.2 Functions Review

Ideas:

- Vertical line test is used for confirming if a the curve is a function. Horizontal is used for testing if curve has an inverse. You should be able to reason why which is which. If not, do it as an exercise.
- Substitution, elimination, and isolation were the workhorses of our toolbelt in the last chapter. That likely won't go away. In particular, we can use substitution to make problems easier. Consider the following AMC problem: Let $f(x^2+1) = x^4 + 5x^2 + 3$. What is $f(x^2-1)$? We can solve it using subtitution. We let $t = x^2 + 1$ and we see if we can isolate from there. We find $t - 1 = x^2 \Rightarrow x^4 = (t-1)^2$. We now have the x terms we need. Namely, we can rewrite $f(x^2+1)$ as $f(t) = (t-1)^2 + 5(t-1) + 3$. We can clearly simplify that down and then substitute $x^2 - 1$ into t to solve the problem.

Problems:

i. Find $g^{-1}(3)$ given that $g(x) = \dfrac{3x+1}{2x+g(x)}$.

ii. For which constants a, b, c and d does the function $f(x) = \dfrac{ax+b}{cx+d}$ have an inverse?

iii. Some problems are just messy

iv. Suppose f and g have inverses and $h = f \circ g$. Show that $h^{-1} = g^{-1} \circ f^{-1}$.

Questions:

1. How teh fricken heck...? This isn't the first time an AoPS nested function problem has seemed to be...well...strange. I don't understand what we're doing or how it helps us solve any problem besides a nested function problem.

   Edit: Lol. When you don't know what functional equations are :P

2. determinant go brrr? Consider the function $f = \dfrac{ax+b}{cx+d}$. In the linked problem, we proved that said function has an inverse if and only if $ad - bc$ is nonzero. I wonder how we could interpret that with linear algebra?

6.3 Complex Numbers

Ideas:

- Complex numbers don't behave *exactly* the same way as arbitrary vectors, but there's lots of overlap. Particularly with respect to distances in the plane. One difference, I believe, is in that of multiplication.

- As we've seen before (translation: I totally forgot because I'm a silly stupid head), we can multiply by the conjugate to rewrite a real number divided by a complex number as just a complex number. For example,

$$\frac{1}{3+2i} = \frac{1}{3+2i} \cdot \frac{3-2i}{3-2i} = \frac{3-2i}{(3+2i)(3-2i)} = \frac{3-2i}{3^2+2^2} = \frac{3-2i}{13}.$$

- When working with fractions, it is good practice to leave our final result in the form

$$\frac{\overline{z}}{z\overline{z}} = a + bi$$

rather than $\frac{1}{z}$.

- Problems with complex numbers and powers should make you think of breaking the problem into several subproblems. Writing out $(x+i)^6$ is terrible...especially because it will be really simple in the end! Instead, compute $(x+i)^2$ and then recognize that $(x+i)^6 = (x+i)^2(x+i)^2(x+i)^2 = (x+i)^4(x+i)^2$.

- For two complex numbers z, w it is always true that $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z} \cdot \overline{w}$. We can use this to speed up problem solving. For example, say you have a problem in which you somehow end up needing to know what

$$|(10+24i)(8-6i)|$$

is. We can use $|zw| = |z||w|$ to avoid expanding the product. For full simplification, see here.

- Two complex numbers are equal if and only if both their real and imaginary parts are equal.

- Working with $|z|^2$ is usually nicer than $|z|$. How can we use $|z|^2$ to eventually solve the following problem? Find the complex number z that satisfies $z + |z| = 2 + 8i$.

- $Re(z) = \frac{z+\overline{z}}{2}$ and likewise $Im(z) = \frac{z-\overline{z}}{2i}$. We can use what we know about Re and Im to solve problems such as Find and graph all such z that satisfy $|z-3| = |z+2i|$. Namely, we write z as $a+bi$, do some algebraic manipulations, and get $6a + 4b = 5 \Rightarrow 6\text{Re}(z) + 4\text{Im}(z) = 5$. It should be fairly clear that we get a line (what does 6x + 4y = 5 look like?) but it may not be clear how we should answer the original question. To answer the question, we write

$$\begin{aligned} 6\mathrm{Re}(z) + 4\mathrm{Im}(z) &= 6\frac{z+\overline{z}}{2} + 4\frac{z-\overline{z}}{2i} \\ &= 3(z+\overline{z}) - 2i(z-\overline{z}) \\ 5 &= (3-2i)z + (3+2i)\overline{z}. \end{aligned}$$

Alternatively, we could right a more intuitive

$$\mathrm{IM}(z) = \frac{5-6\mathrm{Re}(z)}{4}.$$

- For the haters: Hamiltonian representation of complex numbers go brrrr

$$\begin{aligned} (a,b) + (c,d) &= (a+c, b+d), \\ (a,b)\cdot(c,d) &= (ac-bd, ad+bc). \end{aligned}$$

$$\begin{aligned} (3,7) + (-3,4) &= (0,11) \\ (3+7i) + (-3+4i) &= 11i \end{aligned} \qquad \begin{aligned} (3,4)\cdot(-2,1) &= (-10,-5) \\ (3+4i)\cdot(-2+i) &= -10-5i \end{aligned}$$

$$\begin{aligned} (0,1)\cdot(0,1) &= (-1,0) \\ i\cdot i &= -1 \end{aligned} \qquad \begin{aligned} (4,-1)\cdot(4,1) &= (17,0) \\ (4-i)(4+i) &= 17 \end{aligned}$$

Problems:

i. Given a complex number $z = a+bi$, for what values of a, b is $\frac{z}{\overline{z}}$ real? What about imaginary? Note: the fraction is the complex number over its conjugate.

ii. Find an equation whose graph is the perpendicular bisector of the segment connecting $4+i$ and $7-2i$.

iii. Show that the points x, y, w, z are the vertices of a parallelogram in the complex plane if and only if the sum of some two of them is equal to the sum of the other two.

iv. Find all complex numbers z such that $|z-1| = |z+3| = |z-i|$. Solve analytically and geometrically.

v. Two solutions of $x^4 - 3x^3 + 5x^2 - 27x - 36 = 0$ are pure imaginary numbers. Find these two solutions. Two solutions should make you think of quadratics!

vi. Define a sequence of complex numbers by $z_1 = 0$, $z_{n+1} = z_n^2 + i$ for $n \geq 1$. How far away from the origin is z_{111}?

vii. AIME problem.

viii. Bruh waht (challenge problem)

ix. Show that the points w, x, y, and z are the vertices of a parallelogram in the complex plane if and only if the sum of some two of them is equal to the sum of the other two. Treat it like a real proof with lemmas where necessary (you'll need at least one).

Questions:

1. Ahhh this is fun. A few observations. First, we can probably use concepts from geometry to talk about reflection. I imagine the maths would be quite pretty. Second, it seems to me there is nothing special about complex numbers here. Couldn't we say the same for vectors?

2. Will there always be nice symmetry like in this problem? I wish, but I think not. Look at the coordinates. It appears (though I don't know if it's actually true) that the y intercept is (0,2) which means we go over two and up 5 and down 7 and over 3, which is funny, given that $w = 3 - 5i$ and $z = -2 + 7i$

3. We proved that $z\overline{z} = |z|^2$ and $|zw| = |z||w|$. What do those facts mean geometrically?

6.4 Quadratics

Ideas:

- Vieta's formulas state $-\dfrac{b}{a} = r + s$ and $\dfrac{c}{a} = rs$ where r, s are the roots of the quadratic $ax^2 + bx + c$. We can prove Vieta's formulas in at least two ways. We could set expression equal to zero, factor out a, and notice that the formulas follow. Alternatively, we can do clever guess work. If $a = 1$ we have the familiar

  $$x^2 + bx + c = (x - r)(x - s)$$

  so we try

  $$ax^2 + bx + c = a(x - r)(x - s)$$

  and indeed we find the formulas hold again.

- Vieta's formulas give us ideas about how to be clever when factoring. Namely, if we have some $ax^2 + bx + c = 0$ we can recognize that we may want either $(ax - p)(x - q)$ or alternatively some $(Wx - Z)(Yx - V)$. We can then multiply out and figure out what is what.

- Symmetric expressions involving the roots of a quadratic are a clue to try using Vieta's Formulas.

- **Sidenote:** Vieta discovered a wonderful infinite product:
$$\pi = 2 \times \frac{2}{\sqrt{2}} \times \frac{2}{\sqrt{2+\sqrt{2}}} \times \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \times \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}} \times \cdots$$

- The phrase "the zeros" is just another way of saying "the roots".

- When we have information about the sum and the product of two numbers, we can let the numbers be the roots of a quadratic and use Vieta to construct a quadratic whose roots are the numbers. See here.

- **Putnam Concept:** If you don't see how to simplify a problem right away, plugging in a few numbers to try and recognize a pattern is a great tool. Practice it here Algebra go brrrr

- Go back and review discriminants. Main idea: assuming real coeff, $b^2 - 4ac \geq 0 \Rightarrow$ roots are real whereas $b^2 - 4ac < 0 \Rightarrow$ nonreal. Rational occurs when coeff rational and $b^2 - 4ac$ is square of rational number. Also, only one root when discriminant is zero. **Important**, what we said earlier requires nice coeff. if coeff nonreal, then roots need not be a conjugate pair

- We can use Vieta's Formulas to construct quadratic equations in reverse. see here

- If the coefficients of a quadratic are nonreal, the roots need not be a conjugate pair.

Problems:

i. How many integers x satisfy the equation $(x^2 - x - 1)^{x+2} = 1$?

ii. Suppose $f(x) = x + \sqrt{x}$ and $g(x) = x + 1/4$. Evaluate $g(f(g(f(g(f(7))))))$.

iii. Let $f(x) = \dfrac{7x^2 - 4x + 4}{x^2 + 1}$, where the domain of f is all real numbers. Find the range of f.

iv. Do any challenge problem that you cannot immediately think of how to solve (so probably all of them).

v. If $\frac{x^2 - bx}{ax - c} = \frac{m-1}{m+1}$ has solutions for x such that each solution is the negative of the other, then find m in terms of a and b.

EDIT: The challenge problems offer excellent practice in developing mathematical maturity. That is, they appear to be extremely awful, messy problems. Sometimes they are a bit messy, but even then, it's not as bad as you think. Applying strategic thinking will work wonders.

6.5 Conics

Ideas:

- I don't know the technical phrasing off the top of my head, but a parabola has a focus and a directrix. The focus and directrix are oriented such that the focus is on the line of symmetry which goes through the vertex and is perpendicular to the directrix. Then, every point on the parabola is equidistant to the focus and the directrix. Also, the vertex is the midpoint between the directrix and the focus.
- I found section 5.1 to be some of the dryest material I've seen from AoPS. However, even in that material, there was value. They explored the focus using parametrization which makes me wonder if that's why the curve is called a PARAbola. Parametric. Parabola. No?
- Graph a parametric, interactive ellipse in desmos. Use another book. Sadly, AoPS is not always perfect.
- The distance formula (Pythagorean Thm) is one of your strongest tools when rederiving various conics equations.

Problems:

i. Let x and y be real numbers satisfying the equation $x^2 - 4x + y^2 + 3 = 0$. Find the maximum and minimum values of $x^2 + y^2$.

ii. Fantastic AIME problem:

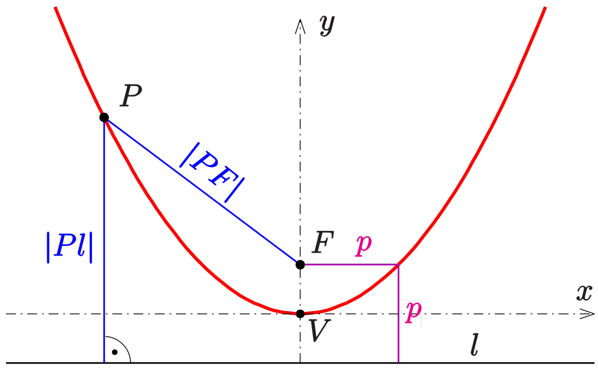
A car travels due east at $\frac{2}{3}$ mile per minute on a long, straight road. At the same time, a circular storm, whose radius is 51 miles, moves southeast at $\frac{\sqrt{2}}{2}$ mile per minute. At time $t = 0$, the center of the storm is 110 miles due north of the car. At time $t = t_1$ minutes, the car enters the storm circle, and at time $t = t_2$ minutes, the car leaves the storm circle. Find $(t_1 + t_2)/2$.

6.5.1 Derivations of Standard Form

Parabola:

We define a parabola to be the curve such that for any point P on the curve, the distance $|PF|$ is the same as the distance $|P\ell|$ where ℓ is the fixed line known as the directrix and F is the fixed point known as the focus. In set builder notation, we write

$$\{P : |PF| = |P\ell|\}.$$



$|PF| = |Pl|$

To derive the equation of a parabola, we use the pythagorean theorem. Let $P = (x, y)$, $F = (h, k+t)$, and $\ell = k - t$. Here, the t is the parameter that adjusts the distance between the focus F and the vertex $V = (h, k)$. Then, via the Pythagorean theorem and the definition of a parabola, we have

$$\sqrt{\big(y-(k-t)\big)^2} = \sqrt{\big(y-(k+t)\big)^2 + (x-h)^2}.$$

If it isn't clear, we are simply using the Pythagorean theorem as the distance formula. The left side denotes $|P\ell|$ and has no x term because the x values are the same. The right side is the distance between P and F. To remove the radicals, we square both sides and rearrange to get

$$\big(y-(k-t)\big)^2 - \big(y-(k+t)\big)^2 = (x-h)^2.$$

From here, many terms cancel each other out. In the end, we get

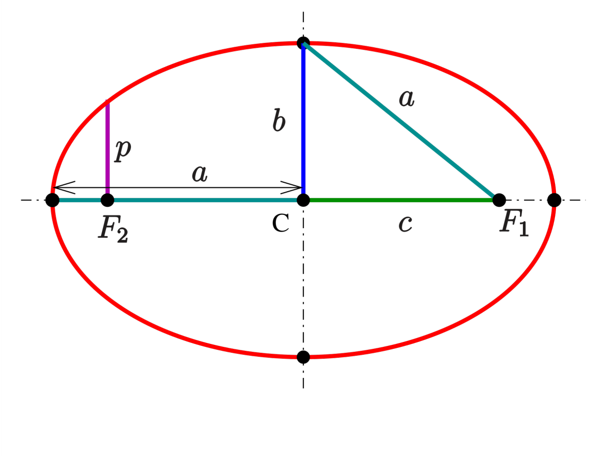
$$4t(y-k) = (x-h)^2$$

which implies

$$y = \frac{1}{4t}(x-h)^2 + k.$$

Ellipse:

We define an ellipse to be the set of points P around two focal points F_1, F_2 such that the sum $|PF_1| + |PF_2|$ is constant.



6.6 Polynomial Division

Ideas:

- For any polynomial $p(x)$, the sum of the coefficients of $p(x)$ is $p(1)$ and it should be noted that this holds for composition of polynomials. If $f(x)$ is some polynomial and $g(x)$ is some other polynomial, $p(x) = f(g(x))$ is still a polyomial so the sum of the coefficients of $f(g(x))$ is simply $f(g(1))$.

  Edit: Given what I found in the text, this concept through me for a loop the first time—it still does. I think the idea is, yes, every polynomial blah blah sum of coeff is 1. Well, let $f(g(x)) = p(x)$. Then, $p(x)$ is the sum of coeff. So what does $p(1)$ mean in terms of $f(g(1))$? Well, x is the input for both polynomials (which are the same), so just put 1 in there and crank it out....you know, it's still not obvious. I should ask a professor.

- I'm not sure when this trick may come up, but it's clever so I'll write it down. Suppose we have some polynomial $p(x)$ and another polynomial $r(x)$. Suppose also that we want to find a polynomial $q(x)$ such that $p(x) \cdot q(x) = r(x)$. Doing so is not as hard as it seems so long as we take it one coefficient at a time. We'll need the right exponents to match up the degrees properly, but other than that, we simply create an easily solvable equation for each coefficient. See here.

- When the polynomial $f(x)$ is divided by the polynomial $d(x)$, the quotient $q(x)$ and remainder $r(x)$ are the polynomials such that

  $$f(x) = q(x) \cdot d(x) + r(x)$$

  and either $r(x) = 0$ or $\deg r < \deg d$.

- We can quickly check our polynomial long division by writing the "polynomial division algorithm" and then setting x to a nice value like 1 or 0. Be careful though, I wouldn't assume that there are no instances where this check can fail.

- Proving uniqueness (in general, not necessarily related to polynomial division) is likely something you will face in the future. We see here how to prove that the quotient and remainder are unique when one polynomial is divided by another.

- Synthetic division is a little gimmicky, but it's faster so it's better for scratch work. Here's the approach. We write the coefficients under a sort of L bracket thingy like so:

$$\begin{array}{r|rrr|r l} 5 & 1 & 7 & -1 & 11 & (\heartsuit) \\ & & 5 & 60 & \mathbf{295} & (\diamondsuit) \\ \hline & 1 & 12 & 59 & \mathbf{306} & (\spadesuit) \end{array}$$

  We write the negation of the constant term on the outside, then we bring down the first value, multiply, place result below, add, bring down, repeat. So, $x^3 + 7x - x + 11$ divided by $x - 5$ is $x^2 + 12x + 59 + \dfrac{306}{x-5}$.

  IMPORTANT: Synthetic Division only works when the coefficient of the linear term in the divisor is 1. There are ways around this. Rederive it yourself as an exercise or see here.

- Chaining together sub problems allows us to use synthetic division when dividing by a non-linear polynomial. For example, say we want to divide some polynomial $f(x)$ by some quadratic polynomial $p(x)$. Factor $p(x)$ into $(x-r)(x-s)$ where r, s are roots and then first divide by $(x-r)$ to get $f(x) = (x-r)q(x) + R_1$ where $q(x)$ is quotient, R_1 remainder. Then, divide $q(x)$ by $(x-s)$ to get $q(x) = (x-s)g(x) + R_2$. Finally, doing some substitution yields

$$f(x) = (x-r)[(x-s)g(x) + R_2] + R_1.$$

- Using what we know about polynomial division, we can prove the **Remainder Theorem**. The Remainder Theorem states that when a polynomial $f(x)$ is divided by $x - a$, the remainder is $f(a)$.

Problems:

i. Let $P(x)$ be a polynomial such that when $P(x)$ is divided by $x - 19$, the remainder is 99, and when $P(x)$ is divided by $x - 99$, the remainder is 19. What is the remainder when $P(x)$ is divided by $(x-19)(x-99)$?

ii. Find the remainder when the polynomial $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 - x$.

iii. When $y^2 + my + 2$ is divided by $y - 1$, the quotient is $f(y)$ and the remainder is R_1. When $y^2 + my + 2$ is divided by $y + 1$, the quotient is $g(y)$ and the remainder is R_2. If $R_1 = R_2$, then find m.

iv. Find the remainder when x^{100} is divided by $(x-1)(x-2)$. You ARE allowed to use perfect powers in your answer.

v. Find the remainder when $x^{100} - 4x^{98} + 5x + 6$ is divided by $x^3 - 2x^2 - x + 2$.

vi. Let $f(x) = x^2 - 3x + 3$ and $g(x) = 3x^3 - 5x^2 - 3x + 12$. The solutions of the equation $f(x) = 0$ are also solutions of the equation $g(x) = 0$. The equation $g(x) = 0$ has a third solution besides the two solutions of $f(x) = 0$; what is this third solution?

vii. Cheeky

viii. When

$$P(x) = x^{81} + Lx^{57} + Gx^{41} + Hx^{19} + 2x + 1$$

is divided by $x - 1$, the remainder is 5, and when $P(x)$ is divided by $x - 2$, the remainder is -4. However,

$$x^{81} + Lx^{57} + Gx^{41} + Hx^{19} + Kx + R$$

is exactly divisible by $(x-1)(x-2)$. If L, G, H, K, R are real, compute the ordered pair (K, R).

ix. AHHHH ROOOKIE MISTAKE. You can't divide by a variable without confirming that that variable can never be zero! Also, don't forget about imaginary numbers!

x. Yeah I have no idea lol

xi. Find the remainder when $x^{81} + x^{48} + 2x^{27} + x^6 + 3$ is divided by $x^3 + 1$.

6.7 Polynomial Roots I

Ideas:

- The Remainder Theorem and The Factor Theorem are similar, but there's a slight difference between the two. The Remainder Thm states that when a polynomial $f(x)$ is divided by $x - a$, the remainder is $f(a)$. The Factor Thm tells us that when $f(a) = 0$, $x - a$ is a factor of $f(x)$.
- Let g, f denote the nonzero polynomials $g(x)$ and $f(x)$. Then, if g is a factor of f, every root of g is a root of f. The converse is not necessarily true. See here.
- Let the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

have roots r_1, r_2, r_3, …, r_n. Then we can write $f(x)$ as

$$f(x) = a_n(x - r_1)(x - r_2) \cdots (x - r_n).$$

- From the above we can observe that a polynomial with degree n cannot have more than n roots.

- If we find that a is a root of the polynomial $p(x)$, we can simplify our search for more roots of $p(x)$ by dividing $p(x)$ by $x - a$. Since a is a root of $p(x)$, this division will leave no remainder, and we will have

$$p(x) = (x - a)q(x).$$

  We can then continue our search for roots of $p(x)$ by searching for roots of $q(x)$.

- If the sum of the coefficients of a polynomial $p(x)$ is zero, then $(x - 1)$ is a factor of $p(x)$. Remembering that is a very useful trick.

- **IMPORTANT:** Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0,$$

  where the coefficients of $f(x)$ are integers. If r is a nonzero integer root of $f(x)$, then $r \mid a_0$. This idea is crucial enough that you should prove it. I've included such an exercise in the problems.

- Graphing is also very helping. Consider some polynomial $f(x)$ and some nonzero real number k. Suppose that $f(0) = k$ and $f(1) = -k$. Via the intermediate value theorem, there must be an x between 0 and 1 such that $f(x) = 0$.

- Combining the last two ideas makes finding roots not too difficult. We first check 1 and -1 as those are easy to verfiy and then we check factors of the constant. If we don't find a root right away, we may find that two of the values we check call for the intermediate value theorem.

- Finding the roots helps us with getting a rough idea of what a graph make look like even for functions that may be otherwise hard to plot. Namely, we can use the roots to determine when the plot is positive or negative. See here.

- **Rational Root Thm:**

  Let $f(x)$ be the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0,$$

  where **all the a_i are integers**, and both a_n and a_0 are nonzero. If p and q are relatively prime integers and $f(p/q) = 0$, then $p \mid a_0$ and $q \mid a_n$.

- Use **Synthetic Division** to speed up your guess work when hunting for rational roots.

- We can decrease our guesswork by looking for bounds. All nonnegative coefficients or alternating coefficients should sound some alarms to check for bounds.

- Let $p_1(x)$ and $p_2(x)$ be polynomials of degree at most n. If $p_1(x) = p_2(x)$ for $n+1$ distinct values of x, then $p_1(x) = p_2(x)$ for all x. We call this the **Identity Theorem** for polynomials.

Problems:

i. The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has roots 1, 3, 5, and 7. Determine all the coefficients of $f(x)$. Don't use a system of equations to solve the problem. That should work, but that isn't the point.

   Alternatively, prove that a polynomial of degree n with a leading coefficient a_n and roots $r_1, \ldots, r_n$ can be written in the form $a_n(x - r_1)\cdots(x - r_n)$.

ii. Suppose k is a root of

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0,$$

   where k is a nonzero integer, and all the coefficients of $f(x)$ are integers. Prove that k must divide a_0.

iii. There are four roots of

$$f(x) = x^4 - 8x^3 + 24x^2 - 32x + 16.$$

   We can easily test to find that $f(2) = 0$. We can then check all the other divisors of 16, both positive and negative, and find that no divisors of 16 besides 2 are roots of the polynomial. Is it correct to deduce that the other three roots of $f(x)$ are not integers?

iv. 7.2.1 and 7.2.2 are good exercises. If you remember how to solve such problems, feel free to skip.

v. Suppose that $f(x)$ is a polynomial with integer coefficients such that $f(2) = 3$ and $f(7) = -7$. Show that $f(x)$ has no integer roots.

vi. Consider the polynomial $g(x) = 12x^3 + 16x^2 - 31x + 10$. Suppose $g(p/q) = 0$, where p and q are integers and p/q is in reduced form. Rewrite $g(p/q) = 0$ using the definition of g. Multiply by by the appropriate power of q to get rid of the fractions. Now, why must q divide 12?

vii. Number Theory assumed I know Rational Roots Theorem lmfao

viii. Prove the Rational Roots Thm

ix. Analysis like AHSME problem

x. To the right is the graph of $y = f(x)$, where $f(x)$ is a polynomial. Explain why the degree of the polynomial is at least 5.

xi. To the right is the graph of $y = f(x)$, where $f(x)$ is a polynomial. The only intercepts of the graph are $(-1, 0)$, $(1, 0)$, $(3, 0)$, and $(0, 3)$, and we have $\deg f < 5$. Find $f(x)$.

xii. Show that if f is a polynomial of degree 4, such that $f(0) = f(1) = f(2) = f(3) = 1$ and $f(4) = 0$, then $f(5) = -4$.

xiii. A certain polynomial $p(x)$ leaves a remainder of a when divided by $x - a$, a remainder of b when divided by $x - b$, and a remainder of c when divided by $x - c$, where a, b, and c are different. What is the remainder when $p(x)$ is divided by $(x - a)(x - b)(x - c)$?

xiv. Solve the system

$$\begin{array}{rrrrr} a+ & b+ & c+ & d = & 1, \\ 8a+ & 4b+ & 2c+ & d = & 16, \\ 27a+ & 9b+ & 3c+ & d = & 81, \\ 64a+ & 16b+ & 4c+ & d = & 256. \end{array}$$

xv. Let $P(x)$ be a polynomial whose degree is 1996. If $P(n) = \frac{1}{n}$ for $n = 1, 2, 3, \ldots, 1997$, compute the value of $P(1998)$.

xvi. Show that if a polynomial $f(x)$ leaves the same remainder when divided by $x - a$ and by $x + a$ for all constants a, then the polynomial does not have any terms with odd degree.

xvii. Show that if a polynomial $f(x)$ leaves the same remainder when divided by $x - a$ and by $x + a$ for all constants a, then the polynomial does not have any terms with odd degree.

xviii. Let

$$P(x) = x^4 + ax^3 + bx^2 + cx + d,$$

where a, b, c, and d are constants. If $P(1) = 10$, $P(2) = 20$, and $P(3) = 30$, compute

$$\frac{P(12) + P(-8)}{10}.$$

xix. In order to get to Mars, you must win a video game. The video game chooses 10 points (a, b), where a and b are single-digit integers, and places a disk with radius $1/3$ on each of the points. You must find a polynomial f such that the graph of f hits all 10 discs. However, you must choose your polynomial before seeing where the disks are. Find a polynomial that guarantees you a trip to Mars.

xx. Find all pairs of nonzero integers (a, b) such that $(a^2 + b)(a + b^2) = (a - b)^3$.

6.8 Polynomial Roots II

Ideas:

- Irrational conjugates are not the most frequently used tool, but when you need them, forgetting them will cause a lot of pain.
- Complex roots come in conjugate pairs. Not particularly suprising if you ask me but there you have it.

  Edit: The above holds when the polynomial has **real coefficients**! We have not looked at complex coefficients. Fortunately, we can sometimes make a clever substitution in which we replace our variable with a function such that the result is a polynomial with real coefficients.
- When factoring a challening polynomial with known values (coefficients aren't variables), when all else fails we can take a guess as to what the factoring might be and use a system of equations to figure out the rest. See problem iv. (Assuming I don't mess up the order lmfao).
- We can extend Vieata's Formulas for a quadratic to an n degree polynomial. For the full explanation, see here. Also, I wrote out a slightly shorter version below the questions portion of my notes for this chapter.
- Just as we used Vieta's Formula's in the past to answer seemingly tricky (but actually easy) questions about the values of the sum of the roots squared ($r^2 + s^2$) or the sum of the inverse of those values ($1/r^2 + 1/s^2$) we can do so with arbitrarily many roots. I've attached a link to a problem in the problems below. It should be problem v. (listed as "Problem 8.20").
- As we saw in intermediate number theory, some problems involve a little bit more than a direct application of Vieta's Formulas. Sometimes we need to use what we know about Vieta's Formula's and combine other techniques like grouping to get new equations that help us solve the problem.
- ...I almost feel silly writing this down. Suppose r is a root of some polynomial $p(x)$. Then, $p(r) = 0$. Okay, yes, this isn't new. However, we can use it to solve unusual problems with roots.
- To make equations with roots, we usually end of writing some equation in form $expression = 0$ and then manipulate it to get $r^k = expression\ of\ r$ and can then use that to simplify. See here.
- Constructing polynomials such that there is a root with radicals that are not powers of a half is probably best achieved with some substitution.

Problems:

i. Find a polynomial with integer coefficients that has $\sqrt{\sqrt{2}+\sqrt{3}}$ as a root.

ii. Suppose $z = a + bi$ is a solution of the polynomial equation

$$c_4z^4 + ic_3z^3 + c_2z^2 + ic_1z + c_0 = 0,$$

where c_0, c_1, c_2, c_3, c_4, a, and b are real constants. Prove that $-a + bi$ is also a solution.

iii. Let

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

and $q(x) = a_mx^m$ be polynomials with real coefficients. Show that $\overline{q(z)} = q(\overline{z})$ for all complex numbers z, $\overline{p(z)} = p(\overline{z})$ for all complex numbers z, and finally, show that if $p(z) = 0$, then $p(\overline{z}) = 0$.

Hint: You may want to recall some important properties of complex numbers from Chapter 3.

iv. Find all values of x such that $x^4 + 5x^2 + 4x + 5 = 0$.

v. Problem 8.20

vi. We've discussed "invisible terms" in previous sections. What is the invisble term in the polynomial $x^3 + ax + b$? Suppose that $a, b \in \mathbb{R}$ and a complex root of said polynomial is $1 + 2i$. What are the other roots and what are a and b?

vii. The roots r_1, r_2, and r_3 of $x^3 - 2x^2 - 11x + a = 0$ satisfy $r_1 + 2r_2 + 3r_3 = 0$. Find all possible values of a.

viii. Consider the following problem: Find $(2 + r)(2 + s)(2 + t)(2 + u)$ if r, s, t, and u are the roots of $f(x) = 3x^4 - x^3 + 2x^2 + 7x + 2$. I claim there are (at least) two methods one could use to solve the problem. Solve the problem using both methods, and don't check your answer for either until you've done both.

ix. Find the two values of k for which $2x^3 - 9x^2 + 12x - k$ has a double root.

x. I stopped leaving hints as the link ref ages ago but this is a particularly hard problem so what the heck. You'll need to make use of the properties of complex conjugates you learned in chapter 3. Failing to recognize that immediately is a clear indicator that you need to revisit that material.

xi. Let r be a root of $x^2 + 7x + 17 = 0$. Compute $(r - 1)(r + 2)(r + 8)(r + 5)$.

xii. Find the polynomial P of smallest degree with rational coefficients and leading coefficient 1 such that $P\left(\sqrt[3]{49} + \sqrt[3]{7}\right) = 4$.

xiii. Find all roots of the polynomial $iy^3 - 8y^2 - 22iy + 21$.

xiv. Prove the "Irrational Conjugates Thm".

Questions:

1. I don't really understand the proof given in the problem where we show that if some complex number r is a double root, then $\overline{r}$ is a double root as well. How can we be so sure that there aren't complex values in $q(x)$? I guess that would need to violate previous steps? Or?

   Edit: I think so, yes. Look at this quote from another question: "When r_i is real, the expression $x - r_i$ is a linear factor with real coefficients. Otherwise, r_i is nonreal, so $\overline{r_i}$ is also a root of $p(x)$, because the coefficients of $p(x)$ are real."

Vieta's Formulas Generalization:

Consider the polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and suppose that $p(x)$ has roots $r_1, r_2, \ldots, r_n$. It is obvious that if $p(x)$ has the roots r_i (i from 1 to n) that $p(x)$ can be written in the form $a_n(x - r_1)(x - r_2)\cdots(x - r_n)$. What's not as obvious, but still not particularly difficult, is seeing what happens in the expansion and how it leads to each a_i. Let's ask ourselves, how can we form a term with a power of $n-1$? Well, we simply choose an r_i and then let every other binomial contribute an x to the product. Since there are n roots, there are n distinct ways to do this and we find that $a_{n-1} = a_n(r_1 + r_2 + \cdots + r_n)$. There's nothing stopping us from doing this for every term. In general, we will find that each term of the expansion is of the form $(-1)^k a_n s_k x^{n-k}$ where s_k denotes the symmetric sum of $\binom{n}{k}$ terms (once for each combination of k of the n roots). Hence, we can generalize Vieta's Formulas with the following:

Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0,$$

and let the roots of $f(x)$ be r_1, r_2, …, r_n. Then, we have

$$\begin{aligned}
s_1 &= r_1 + r_2 + r_3 + \cdots + r_n &&= -\frac{a_{n-1}}{a_n}, \\
s_2 &= r_1r_2 + r_1r_3 + r_1r_4 + \cdots + r_{n-1}r_n &&= \frac{a_{n-2}}{a_n}, \\
s_3 &= r_1r_2r_3 + r_1r_2r_4 + \cdots + r_{n-2}r_{n-1}r_n &&= -\frac{a_{n-3}}{a_n}, \\
&\vdots \\
s_n &= r_1r_2r_3\cdots r_n &&= (-1)^n\frac{a_0}{a_n},
\end{aligned}$$

where each s_k is the symmetric sum of the roots of $f(x)$ taken k at a time.

6.9 Factoring Multivariate Polynomials

Ideas:

- Factor numbers before variables. Or alternatively, if everything is done symbolically, factor coefficients before variables...yeah...
- Sometimes a little number theory (GCD/LCM) can help.
- When dealing with problems like Putnam 2018 A1, moving the multivariate term is doable, but much, much messier. Just move everything else.
- Remember your fundamentals from Ch. 1 (Isolation, Elimination, Substitution, Symmetry, and Systems of Equations)
- Do not assume that problems require finitely many solutions unless it is explicitly stated. Sometimes, solutions to "find all pairs" problems could be of the form (c, v) where c is a constant and v is a variable. Here is an example of one such problem.
- Grouping is a powerful tool when solving Diophantine Equations (equations in which only integer solutions are considered).
- See problem v. If you remember the techniques, it's an idea. If you don't, that's why it's a problem (and an important one, too).
- Our factorization knowledge of sums and differences of powers (see problem v.) can help us with a certain kind of divisibility problem. Namely, if we want to know if a sum/difference of powers is divisible by some integer, we can use the factorization of the sum/difference of powers to check. For example, suppose we wanted to know if $3^5 + 8^5$ is divisible by 11. How does the factorization of $x^5 + y^5$ help us?
- Be very careful to make sure you are solving the problem you are asked to solve, and not the problem given by your own interpretation of the problem statement. For example, "Compute the number of ordered triples such that..." is a different problem statement than "Find each ordered triple such that...". In the case of the ARML problem that inspired this idea, they actually sort of give the same answer, but the given wording makes it more clear when we are finished than the slightly different wording I obtained from misinterpreting the problem.
- Never underestimate the power of adding zero via adding and subtracting the same number.
- I'm not entirely sure the "building a difference of squares" method is important enough for me to deeply understand it, but I'll write it down anyway. The idea is seen in the explanation section I give, the problem following that problem, and an AIME problem.

- When you're stuck, pause, breathe, and clean. Seeing numbers or expressions in a cleaner form can work wonders.
- Multivariate problems such as A1 from Putnam 2019 often require a subtle and counterintuitive trick. We have lots of variables, which isn't pleasant. We need to get rid of some variables, so we...add another variable...but we do so in a clever way! If our added variable can relate the other variables, then we have turned a multivariate problem into effectively a problem of a single variable. A great example of this is Mohamed Omar's Explanation of the Solution to 2019 A1. We also see such a technique used while exploring the Factor Theorem of Multivariate Polynomials.
- When factoring multivariate polynomials, we treat all variables but one as a constant and then factor accodingly until we have the proper degree. When we do so, we don't necessarily need to use $a = b$ or $a = c$ (assume some polynomial $f(a, b, c)$ is what we are factoring). We could try $a = 0$. That is, we can check if a itself is a factor.
- The statment 0=0 is not helpful to us when factoring multivariate polynomails. That is, say we wish to factor some $g(x, y, z)$. Then, suppose we find that $g(x, y, z) = k(x - y)(x - z)(y - z)$ for some constant k. Checking $x = y = z = 0$ is not helpful because if k is zero then g is always zero (which is probably not the case).
- NEED TO ADD THIS IDEA AND GO BACK AND LOOK AT PROBLEMS FROM SECTION

Problems:

i. Factor

$$2x^5 + x^4y - 21x^3y^2 + 16x^2 + 8xy - 168y^2$$

as the product of linear and quadratic polynomials with real coefficients.

ii. Find all values of x that satisfy

$$\frac{6}{\sqrt{x-8}-9} + \frac{1}{\sqrt{x-8}-4} + \frac{7}{\sqrt{x-8}+4} + \frac{12}{\sqrt{x-8}+9} = 0.$$

iii. Factor

$$x^2y + xy^2 + x^2 + 2xy + y^2 + x + y.$$

iv. Find all solutions to the system of equations

$$\begin{aligned} xy + 2x + 3y &= 4, \\ yz - y + 2z &= 5, \\ xz - x + 3z &= 33. \end{aligned}$$

This is an excellent problem for practice with tricky systems of equations.

v. The factorization of the difference of cubes should give you an idea of how to factor $x^n - y^n$. If you don't remember how to connect the two, do it as an exercise. Try some simpler cases first, and then work your way up to the general case. You should also try to factor $x^{2n+1} + y^{2n+1}$ using a similar strategy.

vi. Find $(a/c)^3$ if $\dfrac{1}{a+c} = \dfrac{1}{a} + \dfrac{1}{c}$.

vii. Show that $n^4 - 20n^2 + 4$ is composite for all integers n, where $n > 4$.

viii. If $x^3 = 1$ and $x \neq 1$, find the value of $(1 - x + x^2)(1 + x - x^2)$.

ix. Find all ordered pairs (x, y) such that $x - xy^3 = 7$ and $xy^2 - xy = 3$.

x. Here's a good example of that stupid build the difference of squares method that is anything but intuitive.

Explanation go brrrrrr

I occassionally come across a problem that is strange enough that I cannot understand it with scratch work alone. That is, to understand the problem, I must fully write out the solution as if it were a proof to be submitted to a course. One such problem is:

Find a positive integer n such that $x^4 + x^2 + 1$ is a factor of $x^n - 1$. Next, factor $x^n - 1$ as a difference of squares for your choice of n. Then, factor the two factors you just produced. Finally, factor $x^4 + x^2 + 1$ completely over the real numbers.

First, we know that n must be at least 4, otherwise it is not possible for $x^4 + x^2 + 1$ to be a factor of $x^n - 1$. Next, we know that if $x^4 + x^2 + 1$ is a factor of $x^n - 1$, than there is some nonzero polynomial p such that $p(x^4 + x^2 + 1) = x^n - 1$. Therefore, p must have -1 as the coefficient of the x^0 term. To make our lives simple, it is probably best to let p be a polynomial of the form $x^m - 1$ as anything with more terms will simply recquire more work on our part. We can now multiply everything out and make some conclusions. Namely, we see that

$$(x^m - 1)(x^4 + x^2 + 1) = x^{4+m} + x^{2+m} + x^m - x^4 - x^2 - 1$$

so we can conclude that $4 + m = n$ and $2 + m = 4$ otherwise terms will remain. Thus, $n = 6$ and $m = 2$. We now factor $x^n - 1$ as a difference of squares and get

$$\begin{aligned} x^6 - 1 &= (x^3)^2 - 1^2 \\ &= (x^3 - 1)(x^3 + 1). \end{aligned}$$

We can now use what we know about the sum/difference of powers to factor the above like so:

$$(x^3-1)(x^3+1) = (x-1)(x^2+x+1)(x+1)(x^2-x+1).$$

Our final goal is to factor x^4+x^2+1 over the reals. We already know that $(x^4+x^2+1)(x^2-1) = x^6-1$ so with a little rearranging and some substitution from our previous work, we get

$$\begin{aligned} x^4+x^2+1 &= \frac{x^6-1}{x^2-1} \\ &= \frac{(x^3+1)(x^3-1)}{x^2-1} \\ &= \frac{(x+1)(x-1)(x^2+x+1)(x^2-x+1)}{x^2-1} \\ &= (x^2+x+1)(x^2-x+1). \end{aligned}$$

The dirscriminant of both quadratics above is $\sqrt{1-4}$ so neither have real roots, and we are done.

Perhaps the most confusing part about this problem is the solution given by AoPS. It seems the problem is trying to show us that if we were simply told to completely factor x^4+x^2+1 over the reals, we could effectively do what we did above. However, what we did above was a rather...odd way of achieving such a task. It's highly unlikely I would ever think that way when solving a problem. Instead, we could recognize that x^4+x^2+1 seems like a simple enough expression, and with a little substitution, we may be able to simplify the problem into something we're quite comfortable with. In fact, we can. Letting $x^2 = y$, the expression becomes y^2+y+1 which we recognize from the factoring of the difference of cubes. From there, we can achieve the same steps as what we've done above. Ultimately, when faced with challenging problems that don't allow for an obvious path forward, we must do what we normally do, which is try to represent the problem in another way that allows us to make connections to problems we already know how to solve. From there, we use what we know to generate some new equations/ideas and manipulate those equations/ideas accordingly. A great example of working with what we know to try to make discoveries about what we don't—AND A STRANGELY USEFUL APPLICATION OF THE WHOLE x^n-1 problem—is seen here.

Edit: The difference of squares weird factor trick thing will continue to come back to haunt us. It is perhaps more useful than initially anticipated...sadly.

Another Example Explanation Thingy:

There are (at least) two problems from the section on the factor theorem of multivariate polynomials. Those problems are

1) Factor the polynomial

$$f(a,b) = b^3 - ab^2 - ab - b + a^2 + a.$$

and

2) Factor $f(a, b, c)$. Check your factorization by expanding $f(a, b, c)$ and the factorization you found.

We solved both problems by using what we know about factoring single variable polynomials. That is, we treated all variables except one as a constant and factored accordingly.

In 1) we recognized that if $g(a) = b^3 - ab^2 - ab - b + a^2 + a$ then $g(b) = 0$ so $(a - b)$ is a factor of $g(a)$. Finding the other factor is a simple construction exercise. Since g is quadratic in a, the other factor must be linear, so we need only find the constant that yields the right coefficients when the product of factors is expanded.

In 2) we took a similar approach—except this time, we have more than two variables, so we must treat more than one as a constant. There are $\binom{3}{2}$ ways to do that, so we have 3 cases to consider. When we treat the function as a polynomial in a, we see that $a = -b$ is a root, so $(a + b)$ is a factor. By a similar argument, for a polynomial in b we see that $(b + c)$ is a factor, and finally, for a polynomial in c we have $(c + a)$. Notice, f is a degree 3 polynomial so we have $f = k(a + b)(a + c)(b + c)$. We can then let $a = b = c = 1$ and find $k = 3$.

It should be noted that you really ought to check and make sure that the factorization you find is actually correct by plugging in a few triples.

6.10 Sequences and Series

Ideas:

- We can create equations to help us solve problems with arithmetic sequences by using a **common difference**. That is, if we have a few terms of an arithmetic sequence that are in arithmetic progression (say x, y, z) we can gain an understanding of the sequence by recognizing that $y - x = z - y$. Notice though, this isn't necessarily true if we take random terms of the sequence. That is, if the sequence is $a_1, a_2, a_3, a_4, \ldots$ then $a_2 - a_1 = a_3 - a_2$, $a_4 - a_1 = a_8 - a_5$, and so on, but $a_3 - a_1 \neq a_13 - a_4$.

  Example of this idea in practice

- Write dis stuff down bruhv Also, start citing what inspired the idea that you wrote down. That way, if you can't remember some aspect about the idea, you can go back to the problem and see if you can rediscover the idea.

- Why am I so slow today?

- Pretty sure we know this already but here it is...The average of the terms in an arithmetic sequence is equal to the average of the first and last terms of the sequence.
- Why are you continually trying to break through a wall by using your head instead of asking yourself if a ladder exists?
- This will probably come back as a geometric-arithmetic mean inequality thing or something idk

  As predicted. Truthfully, though, I have no interest in these problems until I go back and organize as I think this is all review so far.
- This idea comes up on a Putnam problem somewhere and I believe Michael Penn called it “standard”. You should know it.

  Here’s some similar stuff
- More ideas that are probably already covered in your exposure of series in the first algebra text
- IMPORTANT: alternating series may also be a geometric series, even if the series “looks weird”. Observe that if the ratio is a negative fraction, the series appears to just be some sort of alternating sum, but it’s more than that.

  In general, the ability to recognize a special kind of series like a geo series is really important. (One showed up on the AMC I took and I did not recognize it. Go back and find it!)
- Recall from one of the previous sections that

  $$r^n - 1 = (r-1)(r^{n-1} + r^{n-2} + \cdots + r^2 + r + 1).$$

  We can use this when thinking about geometric series.
- There was something in the number theory text about powers of 5. I don’t think this was it, but it’s interesting all the same.
- More representation tricks
- Good example of a problem in which we search through what we know to help us try to find a solution.
- Not sure if this will be helpful or not.
- Try the problem first, then, write the idea.
- Using bounds to limit the possible solutions is something we’ve seen before, don’t forget it.
- HA! Don’t forget the sum of coeficients trick.

- Lately you've attempted to solve many problems by finding the values of something that you are not asked to find. If the problem begins to get extremely messy using such a method, you should perhaps consider a more literal approach.

Problems:

i. The roots of $64x^3 - 144x^2 + 92x - 15 = 0$ are in arithmetic progression. Find them. Don't fall for the "everything looks like a nail if all you have is a hammer" trick. If one tool isn't working, use another. We've already seen some really great tools!

ii. Find all values of k such that $x^4 - (3k+4)x^2 + k^2 = 0$ has 4 real roots in arithmetic progression.

iii. Warm up problem

iv. Prove that if a, b, c, and d are four consecutive terms in a geometric sequence, then

$$(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2.$$

This problem is significantly easier than it appears. Don't overthink.

v. The roots of $2x^3 - 19x^2 + kx - 54 = 0$ are in geometric progression for some constant k. Find k.

vi. If P is the product of n numbers in geometric progression, S their sum, and S' the sum of their reciprocals, then find P in terms of S, S', and n. Hint: Closed forms will be helpful. If you don't remember what that means, then no hint for you.

vii. Find the sum of

$$1 + 2 + 3 + 6 + 9 + 18 + \cdots + 729 + 1458.$$

viii. Evaluate

$$\sum_{k=1}^{100} (-1)^k k^2.$$

ix. Fantastic Problem.

x. If the integer k is added to each of the numbers 36, 300, and 596, one obtains the squares of three consecutive terms of an arithmetic sequence. Find k.

xi. Arithmetic Prgoressions and Binomial Coefficients

Explanation Thingy

Solve the following problem:

If a, b, c form an arith progression, and

$$\begin{aligned} a &= x^2 + xy + y^2 \\ b &= x^2 + xz + z^2 \\ c &= y^2 + yz + z^2 \end{aligned}$$

where $x + y + z \neq 0$, prove x, y, z also form an arith progression.

We begin with $a - b = b - c$ which implies $2b - a - c = 0$. It then follows that

$$x^2 - 2y^2 + z^2 + 2xz - xy - yz = 0.$$

We can eventually simplify that to $(x + z)^2 - y(x + z) - 2y^2 = 0$. I got to this point in the problem and then got stuck. I factored the $x + z$ term when I should have made a substitution to recognize that we can actually just factor the whole expression. Namely, let $s = x + z$ and then we have $s^2 - sy - 2y^2 = 0$. We factor said expression and get $(s - 2y)(s + y) = 0$ which, in its original form, is

$$(x + z - 2y)(x + y + z) = 0.$$

We were told $x + y + z$ is nonzero so we must have $x + z - 2y = 0$, so indeed, they form an arith progression.

WIL: When you get stuck, remember the fundamentals (substitution being one of them).

6.11 Identities, Manipulation, and Induction

Ideas:

- We can often relate symmetric expressions with identities.
- Because an identity must be true for all permissible values of the variables in the identity, we can test an identity we develop by substituting convenient values for the variables. (this should already be stated in one way or another somewhere, but add it to the list of fundamentals). For that matter, add pattern checking as well.
- Combinations help us think about exapnsions. Or at least, they help me (the wording of this idea is my own, and it probably could be improved). **EDIT:** Yeah, as I said, it could be improved. Combinations seem to help when we are multiplying two things together (even if the two things have more than two terms). However, with three things, it seems that either my technique must change or it all just falls apart. Try to expand in head using tricks.

- Ratio manipulations can be achieved via brute force, working backwards, setting things equal to a variable, or using clever manipulations (adding 1 in disguise). We see great examples of all of the above in just one problem. NOTE: This idea is not completely flushed out. You should know how to use these ideas in practice and you should be able to prove the generalized results in the solution of the linked problem. Also, there's something sort of similar to substitution. It's a part of the setting things e qual to a variable technque. If we have $a/b = c/d$ and we want some other equation like $a + b/a - b = c + d/c - d$, we could work backwards, or we could let $k = a/b = c/d$ which implies $a + b = bk + b$ which, in combination with some similar rewriting, will allow us to get what we want.

- When using ratio identy tricks, you should start with the identity itself to avoid writing things that just aren't true. That is, if you want to use $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ you need to start from $a/b = c/d$.

7 Intermediate Counting and Probability

7.1 Review

7.2 Sets and Logic

Being that I am almost complete with the math major, this section was light work. However, there was still something worth mentioning. The book defines the set theoretic difference S/A to mean $S - A$. Also, though I can complete this problem, it should be easy enough to do in my head, and it wasn't: Set theory go brrrr

7.3 A piece of PIE

To create a pie from scratch, you must first create the universe (Carl Sagan). PIE (principle of inclusion exclusion) can be understood with venn diagrams. Hence, to create a PIE from scratch, we must first create the universe that the diagram exists in.

Ideas:

- When we need to count a total of some kind and we use combinations, we can often break the problem into chunks. The first combination handles the first chunk, the second handles the second chunk and so on. Note: the last chunk *must* be fixed when we set up the problem this way. Examples: see Sanders family problem and waterballoon fight.

- When faced with counting things that are "at least" something or "one or more" of something, our two main tools are PIE and complementary counting.

- PIE can be thought of as a symbolic representation of venn diagrams. $|A \cup B| = |A| + |B| - |A \cap B|$. Similarly,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

- I don't know if there's a name for this, but in some problems it helps to take advanatage of what I'll call exhaustive reasoning. We place one thing, then we place another thing, and the last thing as no where else to go, so we don't place it; it's placed by default. I find it much more demanding to think that way, but it allows for much nicer computation. Using factorials could very quickly mean working with numbers that are in the hundred millions, the billions, or more. As such, it isn't great.

- To add on to previous point, it's not so much that *order* doesn't matter for some problems as much as it is that *distinguishability* doesn't matter. In the cards problem, we do care about order a little. We care enough to properly count all ways we can arrange the cards when we consider certain groups of cards as blocks (e.g. 4 jacks or 3 spades). Other than that, though, we don't mind. Sort of. Depends on the problem...Oi. Didn't explain well... Remember working through the even digits problem? "How many 9-digit numbers have the property that the product of their first and last digits is even?" In the same way we don't care about the digits after/before/inbetween the specific case we're considering (even in front, back, both), we don't care if the Jack of clubs comes before the Jack of spades. In either case, if there are too many cards together, it's an illegal configuration. Therefore, we need not count all illegal configurations so much as all types of illegal configurations...sorta.

- When counting n digit numbers and their configurations, you must not be tricked. An n digit number must have a nonzero digit in it's leftmost point (01 is a one digit number, don't overcount). As such, it may be helpful to simply list out various forms (forms does not imply all arrangements!) instead of trying to be clever.

- Suppose we have four finite sets A, B, C, D. If we wish to count the elements in *at least* two sets, we start with the sum of the cardinalities of the intersections of two sets. Next, we realize "this sum counts every element that appears in exactly two sets exactly once, but it also counts every element that appears in exactly three sets three times." That is, if x is in $A \cap B \cap C$ then x is in $A \cap B$, and $A \cap C$, and $B \cap C$.

- In one problem we were asked to count the number of ways we can choose 4 vertices of a convex n-gon (where $n > 4$) to form a convex quadrilateral, such that at least 1 side of the quadrilateral is a side of the n-gon. I started the problem correctly, completed a third of the problem, correctly deduced that we must use cases to solve the problem...and then gave up...

  In many of the previous problems, we either used cases or we didn't. I wasn't used to mixing techniques—which, now that I think about it, is funny because we've definitely done that before not just in PIE problems but in ALL problems—so I assumed that my work was wrong. It wasn't. When you solve a problem, you must distinguish between assumptions and conjectures. It is fine to make a conjecture, so long as you don't let it dictate your approach when your approach isn't working. Assumptions, however, must dictate your approach. Therefore, you mustn't make assumptions that you cannot justify. I should have solved the problem, and if I recognized that my assumption wasn't justified, I would have.

- When your stuck on a problem, think about how you could ask the same problem in a different way. What is the problem really asking, and would different words convey the same message? For example, one could write a problem as "15 students are each going to enroll in exactly one of economics, psychology, or sociology. In how many ways can they enroll, provided that no class is left empty?" or one could write, "Consider some set S such that $|S| = 15$. In how many ways can we partition S into 3 nonempty sets?"

Problems:

i. A school with 100 students offers French and Spanish as its language courses. Twice as many students are in the French class as the Spanish class. Three times as many students are in both classes as are in neither class. The number of students taking both classes is even, and fewer than 10 students are in neither class. How many students are taking Spanish?

ii. Scientists sit around a table...

iii. Solve the following problem without a calculator: The Sanders family has 3 boys and 3 girls. In how many ways can the 6 children be seated in a row of 6 chairs, so that the boys aren't all seated together and the girls aren't all seated together?

iv. What is the probability that no 4/3 cards of same rank/suit are together?

v. How many positive integers less than 1000 are relatively prime to both 10 and 12?

vi. Find a "Think about it!" solution to the following problem: Yeechi has a deck of cards consisting of the 2 through 5 of hearts and the 2 through 5 of spades. She deals two cards (at random) to each of four players. What is the probability that no player receives a pair?

vii. Langauges and clever counting...(challenge problem)

viii. Use a combinatorial argument to prove that for any positive integer k less than 9,

$$9^k - \binom{9}{1}8^k + \binom{9}{2}7^k - \cdots - \binom{9}{7}2^k + \binom{9}{8} = 0.$$

Questions:

1. How the heck are we supposed to use combinations in the following problems? I solved it with permutations, but I had to use a calculator as the numbers were too big to work with in a reasonable time via hand.

   The Sanders family has 3 boys and 3 girls. In how many ways can the 6 children be seated in a row of 6 chairs, so that the boys aren't all seated together and the girls aren't all seated together?

   I will work through the solution and try to put it in my own words so I understand it. I believe the idea is that we aren't neglecting order, we simply aren't considering it until the very last minute. In doing so, we simplify the problem. Namely, we start by thinking about indistinguishable boys and girls. So, our combinations are of the form

   $$BBBGGG \quad \text{or} \quad GGBBGB \quad \text{etc.}$$

We use our "exhaustive reasoning" trick to count the combinations with no restrions. Namely, we notice that there are $\binom{6}{3}$ combinations because once we choose the seats for the girls, the seats for the boys are fixed. We now want to get all the illegal combinations as that's easier to count than the legal ones. PIE tells us that we need to consider the combinations in which the boys sit together, the girls sit together, or both. We start with the boys sitting together. Since we are doing this odd permutation/combination wamo blamo thing we need to just...list it out?

7.3.1 Clarification Work

There are a few problems in Chapter 3 that I find require enough scratch work that it's actually beneficial to formally write it all out.

1. Five standard 6-sided dice are rolled. What is the probability that at least 3 of them show a 6?

   There are several things we want to keep track of while solving this problem. Because it's a probability problem, we know our goal is to determine the number of successful outcomes and the number of total outcomes. The total outcomes are easy to count. Namely, we have 6^5 total possibilites. The successful outcomes are more challenging.

   Seeing the phrase "at least" tells us we should use PIE, so we start with the number of ways at least 3 can be 6. There are $\binom{5}{3}$ ways for 3 die to be 6 and the remaining two die have 6^2 possibilites. We now need to ask, have we overcounted anything? (Yes, we have). In particular, let us focus on what happens when 4 out of the 5 die show a 6.

   How many times have we overcounted the occurences of 4 out 5 die showing a 6 in our previous count? Well, we don't know how many times 4 out of 5 die can be a six, but for now, we don't actually care—which is what confused me when I initially thought about this problem. Consider the follwing roll:

   $$6\ 6\ ?\ 6\ 6.$$

   Since we allowed two out of the five dice to be arbitrary in our 3 out of 5 count, the roll above was counted. How many times was it counted? There are $\binom{4}{3}$ ways we can pick 3 of the 4 dice with a 6 to be the three out of five in our three out of five count. Thus, we overcounted each instance of four dice three times. There are $\binom{5}{4} \cdot 6$ ways for at least four dice to show a six, so our total count of successes (so far) is

   $$360 - 3 \cdot 30.$$

   To finish the problem, we need to consider how we count the case when all five dice show 6. Much like the four out of five case, we need to know if we over/under counted. Notice, if the roll is

   $$6\ 6\ 6\ 6\ 6$$

we have $\binom{5}{3}$ ways for 3 out of the 5 dice to represent the three dice in our first count. Hence, we over counted 10 times. In our count of four out five, we over counted 5 times, and then we multipled by -3 to account for the relationship between three and four. So, for each instance of 5 out 5 dice showing a six, we counted that instance 10 - 15 times so we need to add it back 6 times. Consequently, our final count of successful outcomes is

$$\binom{5}{3} \cdot 6^2 - 3 \cdot \binom{5}{4} \cdot 6 + \binom{5}{5} \cdot 6 = 360 - 90 + 6 = 276.$$

Our final answer, then, is that the probability that at least 3 out of 5 dice show a six in a standard roll is $\frac{276}{6^5} = \frac{276}{7776} = \frac{23}{648} \approx 3.55\%$. ■

2. How many positive integers less than or equal to 3150 have at least three different prime factors in common with 3150?

   We begin by factoring 3150 into its prime factorization $3150 = 2 \cdot 3^2 \cdot 5^2 \cdot 7$. Next, we recognize that the solution will be of the form

   $$\left\lfloor \frac{3150}{30} \right\rfloor + \left\lfloor \frac{3150}{42} \right\rfloor + \left\lfloor \frac{3150}{70} \right\rfloor + \left\lfloor \frac{3150}{105} \right\rfloor - ??$$

   where ?? denotes the overcounting of the integers that share four prime factors with 3150. Now, how can we be sure that we correct for the over counting properly? When working with lots of numbers (or even a few big numbers) it can sometimes be humorously difficult to see the obvious. There are

   $$\left\lfloor \frac{3150}{210} \right\rfloor$$

   numbers that share four prime factors with 3150. In other words, all numbers less than or equal to 3150 that share four prime factors with 3150 are the multiples of 210 that are less than or equal to 3150. It should now be clear that the multiples of 210 are counted once in each of the four previous sets. Therefore, our final answer is

   $$\left\lfloor \frac{3150}{30} \right\rfloor + \left\lfloor \frac{3150}{42} \right\rfloor + \left\lfloor \frac{3150}{70} \right\rfloor + \left\lfloor \frac{3150}{105} \right\rfloor - 3 \cdot \left\lfloor \frac{3150}{210} \right\rfloor = 105 + 75 + 45 + 30 - 45 = \boxed{210}.$$

   ■

3. 7 people are having a water balloon fight. At the same time, each of the 7 people throws a water balloon at one of the other 6 people, chosen at random. What is the probability that there are 2 people who throw balloons at each other?

   Like any other probability problem, we count the successful outcomes out of the total outcomes. Since each person can throw a balloon at one of the 6 other people, each person has 6 choices. Thus, there are 6^7 possible outcomes. The question asks for the probability that there are 2 people who throw baloons at each other. Notice, if there are 4 people throwing balloons at each other (2 pairs of 2) then there are 2 people throwing balloons at each other, so what we really want to count is the number of outcomes in which *at least* one pair (2 people) are throwing ballons at each other. Seeing "at least" tells us to use PIE.

Like always, when we use PIE we start with the base level of what we want to count. We want two people hitting each other—that's an odd sentence—so we count the number of ways a pair of 2 can exist.

Clearly, there are $\binom{7}{2}$ possible pairs, but we can't stop there. Every person besides the first two (the two throwing at each other) have 6 choices, so we have $21 \cdot 6^5$ possible outcomes in which *at least* one successful pair exists.

We now come to the fUn part: correcting for under/over counting. How many ways can we choose one pair from two pairs? Two. Hence, when we counted *at least* one pair, we overcounted two pairs. In venn diagram terms, if you count $|A|$ and $|B|$ you count $|A \cap B|$ twice, once in $|A|$ and once in $|B|$. So, we subtract the number of ways two pairs can occur. There are $\binom{7}{2}$ options for the first pair, and $\binom{5}{2}$ options for the second pair. However, we don't want ordered pairs, so we must divide $\binom{7}{2}\binom{5}{2}$ in half. Then, the three remaining people have 6 choices each, so there are $21 \cdot 10 \cdot 1/2 \cdot 6^3 = 105 \cdot 6^3$ possibilities.

We now need to consider the occurences of three pairs. How many ways are there to select one pair from three pairs? Three. How many ways are there to select two pairs from three? Three. Thus, we over counted twice and then subtracted thrice so we have not counted at all. We correct by adding it back. Using a similar argument as before, we must add back $\binom{7}{2}\binom{5}{2}\binom{3}{2}$ divided by 3!. The final person has 6 choices, so we account for that, too. When that's all said and done, our sum of all successful outcomes is

$$21 \cdot 6^5 - 105 \cdot 6^3 + 105 \cdot 6 = 141,246.$$

Therefore, the probability that at least two people throw water balloons at each other is

$$\frac{141,246}{6^7} \approx 50.5\%.$$

■

4. Let S be a set with six elements. Let $\mathcal{P}$ be the set of all subsets of S. Subsets A and B of S, not necessarily distinct, are chosen independently and at random from $\mathcal{P}$. Find the probability that B is contained in at least one of A or $S \setminus A$. (Recall that the set $S \setminus A$ is the set of all elements of S that are not in A.)

   When attempting the problem, I recognized that there must be the same number of ways B could be contained by $S - A$ as there are for B to be contained in A, but ultimately, I failed to recognize how we can constructively count. Namely, I failed to recognize that if we can categorize every possible successful outcome for any element of S, then we have counted what we need. In particular, say that we wish to count the outcomes in which B is contained in A. In that case, we consider some arbitry

$s \in S$ and we recognize that there are 3 possibilities that satisfy $B \subseteq A$: 1) $s \in A$ and $s \in B$, 2) $s \in A$ and $s \notin B$, or 3) $s \notin A$ and $s \notin B$. Hence, there are 3^6 possibilities in which $B \subseteq A$.

We now need to consider all the outcomes in which B is contained both by A and $S - A$. The only way that all of B can be two distinct sets is if B is empty. When B is empty, any subset of A contributes to both counts so there are 2^6 outcomes we must subtract from $2 \cdot 3^6$. We then get a probability of

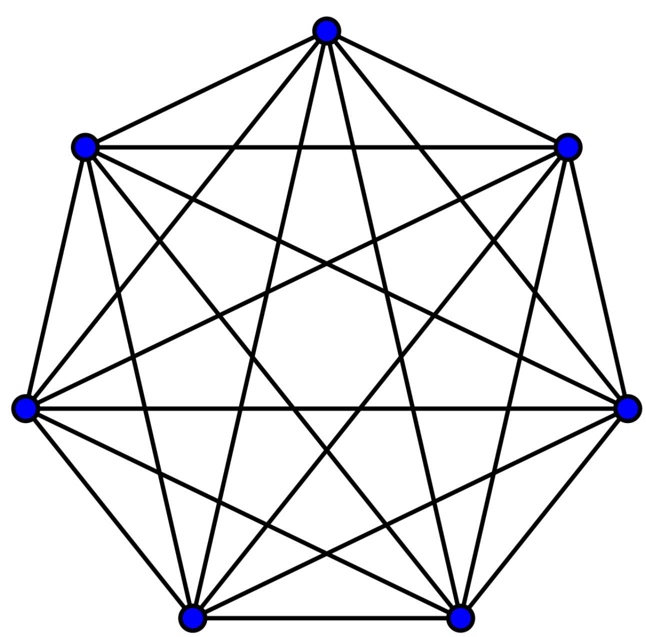
$$\frac{2 \cdot 3^6 - 2^6}{2^{12}} = \boxed{\frac{697}{2048}}.$$

■

7.3.2 A Graph Theoretic Approach To Water Balloons

Problem. Suppose there are seven people in a park and they each throw a water balloon at one of the other six people. What is the probability that at least two people throw a balloon at each other?

Solution. Consider the graph K_7:



Suppose that we make the graph directed by adding two arrows to each edge (towards and away from a vertex incident to the edge). I claim the graph captures all possible throwing combinations. Namely, let each vertex be a person, and let an arc going from some v_1 to v_2

represent v_1 throwing a balloon at v_2. Since the graph was originally K_7, there

$$2 \cdot \binom{7}{2} = 42$$

arcs. An occurence in which two people throw at each other can be thought of as an undirected edge (or a directed but in both directions). There are clearly 21 edges. Enter P.I.E.

We start by counting the instances in which there are at least one pair. There are 21 edges, so there are 21 choices for the pair. Then, the remaining five people can throw balloons at whomever they wish. To represent this, we delete every arc from the two verticies that already have a choice (each other, since they form the pair), so there are...6^5 because there are 6 arcs from each remaining vertex...and it's the same argument. Okay, nevermind.

7.4 Constructive Counting and 1-1 Correspondences

Ideas:

- As we strengthen our ability to count, we should find that we can choose to count either with permutations or combinations. Some problems lend themselves to be better for one than the other. Generally, though, one should try to use combinations if they can as combinations will always be smaller as they do not consider order.

Problems:

i. Consider the set $S = \{1, 2, 3, \ldots, 34\}$. How many ways are there to choose (without regard to order) three different numbers from S whose sum is divisible by 3?

ii. Nine tiles are numbered $1, 2, 3, \ldots, 9$. Each of three players randomly selects and keeps three of the tiles, and sums those three values. Find the probability that all three players obtain an odd sum.

iii. Robert has 4 indistinguishable gold coins and 4 indistinguishable silver coins. Each coin has an engraving of a face on one side, but not on the other. He wants to stack the eight coins on a table into a single stack so that no two adjacent coins are face to face. Find the number of possible distinguishable arrangements of the 8 coins.

iv. What is the sum of all of the 5-digit palindromes?

v. Develop a system to win the reverse keno lottery. Note: The original problem does not provide a solved system. It deals with particular cases.

vi. How many 10-digit numbers have all digits distinct and are multiples of 11111?

vii. In the course of a day, star tennis player Martina Combinova receives 10 different tennis rackets from fans who want her to sign and return the rackets. At various points during the day, Martina takes a break from whatever she is doing to sign some of the rackets. Whenever she decides to sign a racket, she grabs the most recently arrived racket, takes a few strokes with it, then signs it and sends it back. During lunch, Martina's coach tells her that she has signed the 9^{th} racket that arrived. Given that the order of the rackets' arrival is fixed, how many possible post-lunch racket signing sequences are there?

    I'll add this problem to the Putnam practice list, but I will also say here that it likely acts as a great exercise as a general review of combinatorics. If we start from scratch and prove each idea needed to understand the problem, we would be in decent shape to solve many, many other problems. It also introduces the idea of a **stack** (yes, the computer science kind).

viii. We wish to color the integers 1, 2, 3, ..., 10 in red, green, and blue, so that no two numbers a and b, with $a - b$ odd, have the same color. (We do not require that all three colors be used.) In how many ways can this be done?

ix. How many pairs of positive integers (m, n) are there such that the least common multiple of m and n is 21,600?

Calendar

Month 1: December

Dec 1

I worked on Challenge Problems from Chapter 2 today at around 9:30 p.m. Afterwards, I aimed to get through 3.1-3.3 and I've struggled to stay focused. Blah blah this feels stupid. Anyway, when I'm working on a problem and I get stuck, I need to know when to change my strategy. I was working on corrections for Analysis and I suffered from tunnel vision. It resulted in a lot of wasted time and energy. Brain no go squish.

Month 3: February

Feb 4

I solved my first "Find all ordered pairs.." problem!

I solved my first, second, and third HMMT problem!

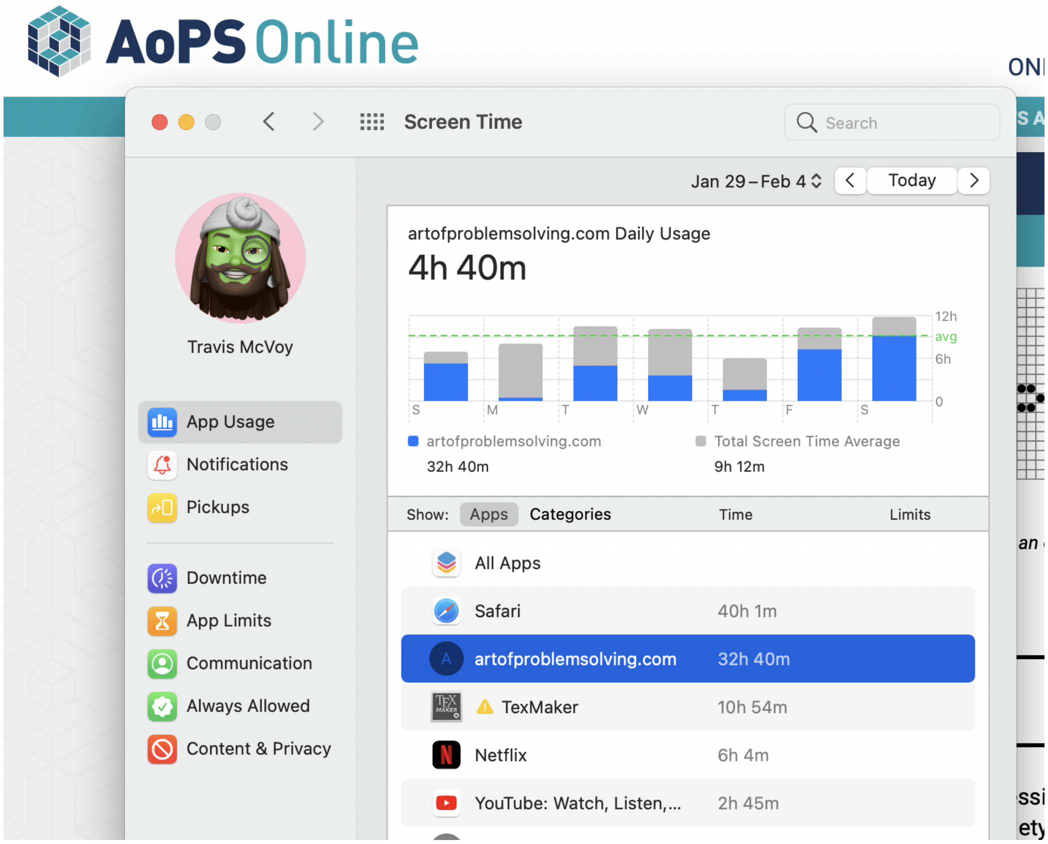
Feb 11

I have missed many problems today, but I solved another HMMT problem!

I passed page 500 in the intro-Algebra book. Consquently, I am completely caught up to the pace I want, and, I have now covered over 1000 pages of material in under 50 days.

Edit: After coming home, I took a break, watched some tv, and got back to work. I set out to complete 80 pages of work today, and I reached that goal.

32 HOURS



Feb 18

Jumped straight to quadratic formula instead of trying other stuff in HMMT problem. In other words, math instincts are getting better :D

Solved one of the hardest problems I've faced (in AoPS) to date. It just so happens it's a HMMT problem. I found a pattern. Proposed a conjecture. Used partial fraction decomposition and then used a telescoping series to solve the problem.

Feb 19

Almost solved my first AIME problem. I had the conceptual understanding almost immediately—which I'm quite pleased with as recognizing patterns in these sorts of problems is usually quite difficult for me. I am disapointed that I messed up the arthimetic which prevented me from solving the problem.

Intro Algebra Finished on Feb 19 @ 10 pm

I believe I started on Jan 29 so that puts completion time at 22 days.

Feb 28

Hit pg. 130 in Intro Counting. When I started, I covered first 27 pgs in a day.

Month 4: March

March 1

Six pages in the morning (approx 90 mintues).

Started day on pg. 130. Ended on 145. Spent roughly 3.5 hours.

March 2

roughly 2 hours. 11...12? pages. 145-156

March 4

I almost finished an AIME problem. Sadly, I recognized all the right ideas but failed to synthesize and ultimately did not get a solution. I still need to work on my resilience. I should have noticed that I hadn't tried my idea out to completion, so I didn't really have a reason to give up. I suppose I did have a reason to work with more paper. Good tip to remember in the future.

March 5

I came up with a different solution to an AoPS problem. We are asked to find a formula for the sum of squares; we were asked this in the hockey stick identity section, so naturally, I converted the problem into a problem with combinations. Namely, I looked at

$$1\binom{1}{1} + 2\binom{2}{1} + 3\binom{3}{1} + \cdots n\binom{n}{1}.$$

We can apply the hockey stick identity to this problem and get a rather nice iterative solution. That is, at each step, we get the combination $\binom{n+1}{2} - \textit{correction terms}$. At each step, we use the hockey stick identity to calculate the correction terms, then, at the very end, we group the correction terms together, use the hockey stick identity one last time, and get

$$\sum_{i=1}^{n} i^2 = n\binom{n+1}{2} - \binom{n+1}{3} = \frac{n(n+1)(2n+1)}{6}.$$

Edit (03/28/23): I'm not sure the above solution is right. The n term outside the first combination worries me...but I put it in to Wolfram and the solutions seem to match up. That's absurd. How tf did I do that?? That's absurd. That works. I have no idea how I recognized the steps during the problem but here's what I figured out now:

$$\begin{aligned} n\binom{n+1}{2} - \binom{n+1}{3} &= \frac{n(n+1)(n)}{2} - \frac{(n+1)(n)(n-1)}{6} \\ &= \frac{3n - (n-1)(n)(n+1)}{6} \\ &= \frac{(2n+1)(n+1)(n)}{6}. \end{aligned}$$

Attempted first USAMO problem. I got...mmm one third of the way there. The question is: 6 points are placed evenly around a circle as in Figure 15.2, and are labeled A, B, C, D, E and F at random. What is the probability that the two triangles ABC and DEF do not overlap? I recognized that there are $\binom{6}{3}$ ways of picking triangles. Three out of the six points, in any order ($ABC = CBA$), go to the first triangle, and the remaining three go to the second triangle. I recognized that the triangles will overlap in every occurrence except when the two triangles are on opposite halves of the circle. What I failed to do is count the valid triangles—which is a bit sad, considering that's the only step left. I was confused about rotation. I don't really see why we should be counting a 60 degree rotation as a different occurence. I guess the idea is that we've accounted for symmetry with the combinations? Maybe?

March 11

I covered 50% of Intro Number Theory in 6 days.

March 16

Solved another HMMT problem. Almost finished with Intro NT.

Solved my first AIME problem! I don't know how the difficulty of AIME works, but I'm willing to be its progressive with the problems. As such, I expect the linked problem to be one of the earlier ones as it was quite easy. A bit disappointing that my first AIME problem was a number theory problem, but oh well.

Note: This was not the first AIME problem I've attempted. I know I've tried at least two more, but it might be higher. If the cube color probability problem is AIME, then I've tried at least three.

March 26

I haven't been as good about logging—in part because I've had to spend a little more time focused on school and research. Geometry is...fine. I like construction. Everything else is a bit annoying. I'm at the end of chapter 5 (pg 120ish). I checked up on intermediate number theory and did the diagnostics problems. In search for some review material, I got distracted and solved my second AIME problem.

March 30

I have decided that I should do a little bit of geometry each day while simultaneously doing other books. Trying to cram all that geometry in at once is just too much. I'm rushing through the material in a way that makes me worried none of it will stick, and I can't risk spending time on something that I can't retain.

8 Putnam Notes (AoPS)

Some things to remember or check for each problem:

1. Have I read the whole problem?
2. Is the answer I found actually what the problem is asking for?
3. Does the problem ask for anything I haven't found?
4. Notes about dividing by zero, zero isn't positive, don't forget to check if zero is an easy solution etc.
5. Any constants that can be removed by factoring probably should be removed.
6. When working with fractions, putting ~~fractions~~ *all terms* into common denominators is usually smart.
7. Graphing, even if it's a sketch, can save a lot of trouble in the right circumstances
8. If you think an idea is right, but you aren't sure how to prove it, right down the idea and write down the fundamentals corresponding to that idea
9. Remember Jeremy. We need to maximize our simplification at every step we take as often as we can. If there's something that would magically make everything better, try it! Don't assume it won't work.

8.1 Practice problems from AoPS:

1. Michael Penn Units Digit

2. Bruh

3. The polynomial
$$f(x) = x^4 + 6x^3 + ax^2 - 54x + c$$
has four real roots r_1, r_2, r_3, and r_4 such that $r_1 + r_2 = 0$ and $r_4 - r_3 = 4$. Find a and c.

4. Let $f(x) = x^2 + (n - a)x + a$, where a and n are integers such that $1 \le a < n \le 49$. For how many values of n are there no possible values of a such that $f(x)$ has rational roots?

5. In the course of a day, star tennis player Martina Combinova receives 10 different tennis rackets from fans who want her to sign and return the rackets. At various points during the day, Martina takes a break from whatever she is doing to sign some of the rackets. Whenever she decides to sign a racket, she grabs the most recently arrived racket, takes a few strokes with it, then signs it and sends it back. During lunch, Martina's coach tells her that she has signed the 9^{th} racket that arrived. Given that the order of the rackets' arrival is fixed, how many possible post-lunch racket signing sequences are there?

6. Find a polynomial $f(x)$ of degree 5 such that $f(x) - 1$ is divisible by $(x - 1)^3$ and $f(x)$ is itself divisible by x^3.

7. Let $P(x) = (x - 1)(x - 2)(x - 3)$. For how many polynomials $Q(x)$ does there exist a polynomial $R(x)$ of degree 3 such that $P(Q(x)) = P(x) \cdot R(x)$?

8. Prove that among any 51 integers, there are two whose squares have the same remainder when divided by 100. Hint: it may be useful to use an idea that has not been covered in intro NT

9. Find the units digit of 7^{7^7}.

10. I'm not a big fan of number theory, but the following problem is actually decent proof writing practice: How many three-digit integers are multiples of both 4 and 6 and have a units digit of 2?

11. I got the right answer, but I think it would be difficult for me to put my answer into words—which as we know, is absolutely necessary. Thus, this problem serves as full justification practice.

12. Bogus NT problem. I find number theory to either be extremely fun or extremely awful. This problem was the first time that's sort of in between. I think we can solve it using systems of equations. That's not how the book solved it, but I never would have gotten that. Also, it's apparently a AHSME problem. I would like to know if that means the answer is in multiple choice form.

13. AIME Problem: How many positive integers have exactly three proper divisors, each of which is less than 50? Failed first attempt (need to put in diligence log)

14. Easy practice with full justification while maintaining efficient problem solving; use variables and logic, *not* guess and check

15. Good practice with inductive reasoning and recognizing that what seems to be a hard problem isn't actually that hard

16. *Extreme Challenge:*

    Prove, in multiple ways, that the number of ways to distribute n pieces of candy to k kids is
    $$\binom{n+k-1}{k-1.}$$
    You may **not** use the stars and bars method in any of your proofs. You should assume that you give out all pieces of candy. That is, you should not include giving no candy to all kids as a legal distribution.

    For a further challenge, clean up your results and prove the hockey stick identity. Finally, relate your two results and describe how the two problems are really the same.

17. Practice for combinatorial proof

18. A line segment is broken at two random points along its length. What is the probability that the three new segments form a triangle? Good practice for full justification.

19. Not a particularly hard problem, but good practice for writing a complete argument as opposed to "it's bloody obvious mate": Let $\overline{CD}$ be a line segment of length 6. A point P is chosen at random on $\overline{CD}$. What is the probability that the distance from P to C is smaller than the square of the distance from P to D?

20. Difficult probability problem from the Mandelbrot Competition.

21. Difficult counting problem from AIME

22. AIME's problem about Pascal's Triangle is good practice for mathematical resiliance.

23. Remember, when you practice, you need to manage your speed. Read the problem. Make sure the problem you solve is the problem you are given.

24. Once you have found a solution, prove it is the only solution.

25. https://artofproblemsolving.com/ebooks/intro-algebra-ebook/par/110137

26. Very easy warm up problem; no intuitive explanations! FULL RIGOR ONLY! (start with what's true)

27. How can we prove that 4 is the *only* solution?

28. Explain the shape of the graph $f(x) = |x-3| + |x+2|$ without actually graphing. You may graph at the end to check your answer.

29. This one is a doozy. If you wanted to do it without a calculator you probably need the intermediate val thm and squeeze thm and some calc. Come back to it later. DOOZY Warning, even then, I'm not entirely sure it's plausible.

30. Practice with sequences and series

31. Very good exercise for the development of mathematical maturity

Examples of full justification:

1. A rather annoying problem: Let $f(x)$ be a polynomial with rational coefficients. If $\sqrt[4]{2}$ is a root of $f(x)$, must $-\sqrt[4]{2}$ also be a root of $f(x)$?

2. Suppose that a function $f(x)$ is defined for all real x. How can we obtain the graph of $f(|x|)$ from the graph of $f(x)$?

3. Adam and Ben start their new jobs on the same day. Adam's schedule is 3 workdays followed by 1 rest day. Ben's schedule is 7 workdays followed by 3 rest days. On how many of their first 1000 days do both have rest days on the same day?

4. Let m be an integer greater than 101. Find the number of values of m such that 101 is its own inverse modulo m.

5. Once you've found a solution, try to prove your solution is the only solution.

6. Don't forget to justify every step (have to actually confirm range via relationship with domain)

7. The order that we multiply matters. If we multiply in the right order, we can divide by without worrying about dividing by zero.

8. Don't forget to read and underline :)

9. Closest you've been to solving a problem that you couldn't solve first try. You asked what would make the problem easy, you had an answer, and you didn't use it. You don't actually need to prove anything exact, you just need to prove that you're close enough, which you can!

10. You had a conjecture which is probably correct and likely one you could prove. It lead you to the right solution, and it did so in a way almost identitical to the provided solution. During your solution, you recognized that you haven't proved your conjecture but you used it anyway. That will likely get you into trouble on the Putnam. Be better! Problem seen here: swag

11. Warm up problem: Find A and B when
$$\frac{2x}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2}.$$
Extra challenge: Use two different methods to solve the problem.

12. Prove that the composition of functions is commutative.
13. warm up counting problem
14. Good practice explaining argument

8.2 AoPS Geometric Problems

Some AoPS problems are geometric but are solved much easier analytically than visually. That sort of thinking is quite difficult for me. I will collect problems that could help me strengthen that skill (and other geometric thinking) right here:

1. Describe the behavior of the graph of $y = 3\left(x - \frac{1}{3}\right)(x+2)(2x-3)^2$ without using a table (note: building the table in your head violates the prompt).
2. How many interior diagonals does an icosahedron have? (An icosahedron is a 3-dimensional figure with 20 triangular faces and 12 vertices, with 5 faces meeting at each vertex. An interior diagonal is a segment connecting two vertices which do not lie on a common face.)
3. See problem iii. in 5.2. (I assume this is a problem from the geometry text)
4. Count the number of ways we can choose 4 vertices of a convex n-gon (where $n > 4$) to form a convex quadrilateral, such that at least 1 side of the quadrilateral is a side of the n-gon.
5. In Problem 10.20, we showed that the sum of the exterior angles of a triangle is 360°. Explain why the sum of the exterior angles of any convex polygon must be 360°.
6. What is the probability that if three points are chosen at random on the circumference of a circle, then the triangle formed by connecting the three points does not have a side with length greater than the radius of the circle?
7. Simpler version of a Putnam A6 question.
8. I don't think this is quite as hard as many of the problems listed here, but I missed part of it the first time around and I really shouldn't have.
9. How many points are common to the graphs of the two equations below?

$$(x-y+2)(3x+y-4)=0$$
$$(x+y-2)(2x-5y+7)=0$$

10. 2 diagonals of a regular heptagon (a 7-sided polygon) are chosen. What is the probability that they intersect inside the heptagon? Extra challenge: prove the lemma necessary for this problem.

11. I mean, if one doesn't know what a decagon looks like I fail to see how this is a plausibly solvable problem without graph paper or a protractor or something. I got the interior angle right, but other than that, I made no progress.

12. Devlop a formula for the number of regions we can make using n lines. Do the same with m circles. If you really want a challenge, consider n lines and m circles.

13. Number of planes that pass through at least three verticies of a cube

14. Cubes, painted faces, and probability

15. How many diagonals of a regular octagon are not parallel to one of the sides?

16. Consider a regular octagon. How many triangles can be formed whose verticies are the verticies of the octagon?

17. Nine lines are drawn in a plane. What is the largest possible number of points in the plane at which at least two of the nine lines intersect?

18. Count all the triangles. Better yet, for some shape in the form of the problem, generalize the number of triangles that can occur with...*arbitrarily many somethings* (don't want to spoil the problem)

19. Problems I solved but are still worth collecting:

    (a) Five different circles are drawn in a plane. What is the maximum number of different points at which the circles can meet?

    (b) Nine parallel lines in a plane intersect a set of n parallel lines that go in another direction. The lines form a total of 360 parallelograms, many of which overlap each other. Find the value of n.

8.3 Beautiful Problems (AoPS, Putnam, or other)

1. An Unexpected Pythagorean Triple

2. The solution to the following Olympiad problem is one of the most elegant works of thought I have ever had the pleasure of reading:

   Determine all non-negative integral solutions $(n_1, n_2, \ldots, n_{14})$ if any, of the Diophantine equation

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1599.$$

3. Find positive integers (a, b, c, d) such that

$$a + \frac{1}{b + \frac{1}{c + \frac{1}{d}}} = \frac{931}{222}.$$

Beautiful is a stretch. Certainly a very playful problem.

9 Past Putnams

9.1 83rd: 2022

9.1.1 A1

Problem statement:

Determine all ordered pairs of real numbers (a, b) such that the line $y = ax + b$ intersects the curve $y = \ln(1 + x^2)$ in exactly one point.

Tools:

Graphing

If we can develop a sense of what the graph looks like, we are well on our way to solving the problem. Notice, $1 + x^2$ is positive for all x. Therefore, $\ln(1 + x^2)$ is smallest when $x = 0$. Next, the squared term will indicate to us that there is some nice symmetry. If you are not convinced, let's plot some points. I will choose nice points, but really, the points themselves don't matter so much; we really only care about the shape of the graph. Choose

$$x_0 = 0, \quad x_1 = \pm\sqrt{e-1}, \quad x_2 = \pm\sqrt{e^2-1}.$$

That will give you a good idea of shape of graph (sort of an upside down bell curve).

Once we have seen the graph, we notice that $y = ax + b$ is linear, so we are effectively looking for places where a line is tangent to the graph. Mmmm no, that's not right either.

Anyway, I need to leverage calculus more. The Intermediate value theorem comes up. Mean value theorem comes up.

$A2$ is a counting problem