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18.440: Probability and Random Variables
Problem Set 2

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Correctness Statement

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Note: The majority of the below problems are from *A First Course in Probability* 8th ed. by Sheldon Ross.

1. **Problem 25.** A pair of dice is rolled until a sum of either 5 or 7 appears. Find the probability that a 5 occurs first. *Hint:* Let E_n denote the event that a sum of 5 occurs on the nth role and no 5 or 7 occurs on the first $n-1$ rolls. Compute $P(E_n)$ and argue $\sum_{i=1}^{\infty} P(E_i)$ is the desired probability.

   Solution. There are four ways to achieve a sum of five with two dice:

   $$(1,4)\ (4,1)\ (2,3)\ (3,2).$$

   The probability that a sum of five occurs is therefore 1/9. By a similar argument, the probability that a sum of seven occurs is 1/6. Therefore, the probability of E_n is

   $$(1 - 1/9 - 1/6)^{n-1}(1/9).$$

   The sum $\sum_{i=1}^{\infty} P(E_i)$ is then written as

   $$\frac{1}{9}\sum_{i=1}^{\infty}\left(\frac{26}{36}\right)^{i-1}$$

   which converges to

   $$\frac{1}{9}\cdot\frac{1}{1-26/36} = \frac{1}{9}\cdot\frac{36}{10} = 0.4$$

   because it's a geo series with a ratio less than 1 in magnitude.

   Now, while the number I got does match the number in the back of the text, we aren't done. We haven't argued that the sum is the desired probability, and I'm very frustrated to say that I don't quite know how. Proposition 6.1 seems as though it would be helpful, but I'd be lying if I said I fully understood it. For now, I'm just going to have to move on, but this **needs to be revisited.**

2. **Problem 48.** Given 20 people, what is the probability that, among the 12 months of the year, there are 4 months containing exactly 2 birthdays and 4 months containing exactly 3 birthdays?

   Solution. There are at least two ways of interpreting the problem statement. Either two and only two birthdays occur in a four month period, three and only birthdays occurr in a different four month period, the rest occurr in the remaining four months. The other interpretation (which I believe to be the correct one) is that four distinct months each have two birthdays and four distinct months each have three birthdays, for a total of twenty. I will write up the second interpretation as it's much easier to solve. Also, though I don't think birthdays actually are normally distributed, I will assume they are within this problem (which means every outcome in our sample space is equally likely).

   To calculate the sample space, we recognize that each of the 20 people could have 1 of 12 possible months for their birthday month (which implies there are 12^{20} possibilities in the space). The desired outcomes will be a bit more work.

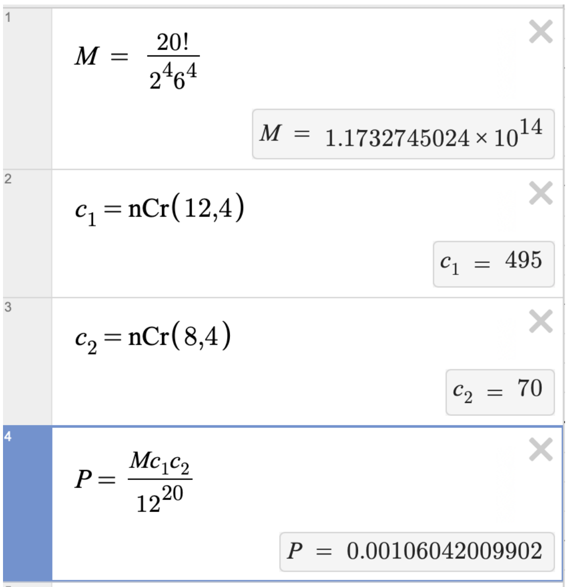
   There are

   $$\binom{20}{2,2,2,2,3,3,3,3}$$

ways to divide the group into the four groups of two and three. For each of those divisions, there are $\binom{12}{4}$ ways to select the four months with (WLOG) 2 birthdays. For each of those, there are $\binom{8}{4}$ ways to select months with three birthdays. Thus, there are

$$\binom{20}{2,2,2,2,3,3,3,3}\binom{12}{4}\binom{8}{4}$$

desired outcomes. If we plug everything into desmos, we find the probability to be roughly 0.00106:



3. **Problem 49.** A group of 6 men and 6 women is randomly divided into 2 groups of size 6 each. What is the probability that both groups will have the same number of men?

   Solution. If both groups have the same number of men, then each group has 3 men. There are $\binom{6}{3}$ ways to choose 3 men to go into the first group. The three remaining men go into the other group by default. For each of those selections, there are $\binom{6}{3}$ ways to choose 3 women to join the men in the first group. There are therefore $\binom{6}{3}^2$ desired outcomes. Since each group must have six people, the number of outcomes in the sample space is given by $\binom{12}{6}$. Assuming equally likely outcomes, the probability that each group has equally many men is

$$\frac{\binom{6}{3}^2}{\binom{12}{6}} = \frac{400}{924}.$$

4. **Theoretical Exercise 10.** Prove that $P(E \cup F \cup G)$ is equivalent to

$$P(E) + P(F) + P(G) - P(E^cFG) - P(EF^cG) - P(EFG^c) - 2P(EFG).$$

   Skipped. This is an inclusion exclusion exercise. If you draw out the venn diagram, the $P(E^cFG)$, $P(EF^cG)$, $P(EFG^c)$, sections are counted twice (so we must subtract them once) and the $P(EFG)$ section is counted three times (so we must subtract it twice). I don't really consider that a full answer, but that's the general idea.

5. **Theoretical Exercise 15.** An urn contains M white and N black balls. If a random sample of size r is chosen, what is the probability that it contains exactly k white balls?

   Solution. There are clearly $\binom{M+N}{r}$ different selections that could occur, so that constitutes our sample space. Then, there are $\binom{M}{k}$ ways that k of the r balls are white. For each of those ways, there are $\binom{N}{r-k}$ ways for the remaining $r-k$ balls to be red. Hence, there are $\binom{M}{k}\binom{N}{r-k}$ desired outcomes so the probability that k of the r balls are white is

$$\frac{\binom{M}{k}\binom{N}{r-k}}{\binom{M+N}{r}}.$$

6. **Theoretical Exercise 20.** Consider an experiment whose sample space consists of a countably infinite number of points. Show that not all points can be equally likely. Can all points have a positive probability of occurring?

   Solution. If all points are equally likely, than each point has probability

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

   But if each point has probability 0, then the sum of all events (sample space) is 0, which violates the normalization axiom (the probability of the sample space is 1). I think it might be possible that every point has positive probability. Suppose each point's probability is given by a term in the geometric sequence in which the first term is 1/2 and each subsequent term is divided in half. That is,

$$a_1 = 1/2, a_2 = 1/4, \ldots, a_n = (1/2)^n.$$

   Then, each term is positive, as none of those terms will ever actually be zero but the sum of all countably inifite probabilities is 1. Therefore, normalization is satisified. My argument seems sound to me, but **this is worth asking about**.

7. A deck of cards contains 30 cards with labels 1, 2, ..., 30. Suppose that somebody is randomly dealt a set of 7 cards of these cards (numbered with seven distinct numbers).

   a) Find the probability that 3 of the cards contain odd numbers and 4 contain even numbers.

   b) Find the probability each of the numbers on the seven cards ends with a different digit. (For example, the cards could be 3, 5, 14, 16, 22, 29, 30.)

   Solution.

   a) The card dealings are all equally likely, so we need only count the sample space and all desireable outcomes. The sample space is given by $\binom{30}{7}$. As for the desireable outcomes, there are 15 odd numbers and we need 3 of them, which can be done in $\binom{15}{3}$ ways. For each of those ways, there are $\binom{15}{4}$ ways to get four even numbers, so the probability that the seven cards contain 3 odd and 4 even is

$$\binom{15}{3}\binom{15}{4}\binom{30}{7}^{-1} = \frac{455 \cdot 1365}{2,035,800} \approx 0.31.$$

b) The sample space is still $\binom{30}{7}$. There are no restrictions on the first card, so there are 30 possible options. Once the first card is selected, two other cards are no longer allowed (say 27 is selected, now 7 and 17 aren't allowed and so on). This leaves 27 options for the second card. Once the second card is chosen, we again cannot choose the remaining two cards with the same last digit. We can continue in this manner and get

$$\prod_{i=0}^{6}(30-3i)$$

but we still aren't done. That count is far too high, and for good reason. Suppose we get dealt 1, 12, 23, 4, 15, 26. The above number counts all 7! ways in which that hand be permuted, which is not what we want. Therefore, the desired probability is

$$\frac{\prod_{i=0}^{6}(30-3i)}{\binom{30}{7}7!} \approx 0.13.$$

8. *Omitted* This problem was not graded in the original problem set, so I'm skipping it to save time.