

# 18.440: Probability and Random Variables

## Problem Set 1

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While there is always a chance of errors in any body of the work, everything I achieve here will be self taught—which means the risk of errors is perhaps higher than usual. If, for some reason, one finds themselves on my page tempted to use my work, please first check that it is indeed correct. If you find an error, email me and I’ll fix it :)

Note: The majority of the below problems are from [A First Course in Probability 8th ed.](#) by Sheldon Ross.

1. **Problem 10:** In how many ways can 8 people be seated in a row if
- there are no restrictions on seating arrangement?
  - persons  $A$  and  $B$  must sit next to each other?
  - there are 4 women and 4 men and no 2 men or 2 women can sit next to one another?
  - there are 5 men and they must sit next to each other?
  - there are 4 married couples and each couple must sit together?

**Solution.**

- There are  $8!$  ways to arrange 8 distinguishable people. If the people are indistinguishable, there is only  $\binom{8}{8} = 1$  way to seat everyone.
- Imagine  $A$  and  $B$  as one unit, then there are  $7!$  factorial ways to sit the 6 people and the single unit. The unit can be flipped, so there are  $2 \cdot 7!$  ways in total. If everyone besides  $A$  and  $B$  are indistinguishable to one another, there's only  $7 \cdot 2 = 14$  ways to do arrange everyone.
- Notice that if no two men are sitting together, we have one of the two arrangements:

$m w m w m w m w$

or

$w m w m w m w m.$

Hence, if the men and women are not distinguishable among each other, there are two ways to seat everyone. If everyone is distinguishable, then there are  $2 \cdot 4!^2$  ways to seat everyone.

- We continue to use the grouping together as a unit idea. There are  $4!$  ways to sit the 3 women and the unit of men. The unit of the men has  $5!$  arrangements, so there are  $3! \cdot 5!$  arrangements.
- Again, imagine each couple as a single unit. Then there are  $4!$  ways to arrange the couples, and each couple can be flipped. Thus, there are  $2^4 \cdot 4!$  ways to seat everyone, assuming each couple is distinguishable from one another (otherwise there are only  $2^4$  ways to sit everyone).

2. **Problem 26:** Expand  $(x + 2y + 3z)^4$ . Note: the original problem uses  $x_1, x_2, x_3$ , but I can't be bothered to type out the subscripts.

**Solution.** We can use the multinomial theorem. I wrote a script to get the coefficients:

```
for i in range(n, -1, -1):
    for j in range(n-i, -1, -1):
        k = n-i-j
        print(" ", i, j, k, end=" ")
        print(" Multinom: ", end="")
        divisor = math.factorial(j) * math.factorial(k) * math.factorial(i)
        print(math.factorial(n) / divisor)
```

which gives the following output:

4	0	0	Multinom:	1.0
3	1	0	Multinom:	4.0
3	0	1	Multinom:	4.0
2	2	0	Multinom:	6.0
2	1	1	Multinom:	12.0
2	0	2	Multinom:	6.0
1	3	0	Multinom:	4.0
1	2	1	Multinom:	12.0
1	1	2	Multinom:	12.0
1	0	3	Multinom:	4.0
0	4	0	Multinom:	1.0
0	3	1	Multinom:	4.0
0	2	2	Multinom:	6.0
0	1	3	Multinom:	4.0
0	0	4	Multinom:	1.0

We then obtain the following expansion of  $(x + 2y + 3z)^4$ :

$$x^4 + 4x^3(2y) + 4x^3(3z) + 6x^2(2y)^2 + 12x^2(2y)(3z) + 6x^2(3z)^2 + 6x(2y)^3 + \dots + (3z)^4$$

which I have no desire to actually simplify and write out.

**Edit:** Either the text or the course notes provides a similar example that I imagine we were to repurpose. Don't really remember where, but I probably made this problem more difficult than it needed to be.

3. **Problem 32:** An elevator starts at the basement with 8 people (not including the elevator operator) and discharges them all by the time it reaches the top floor, number 6. In how many ways could the operator have perceived the people leaving the elevator if all people look alike to him? What if the 8 people consisted of 5 men and 3 women and the operator could tell a man from a woman?

**Solution.** The first part of this problem is effectively a rewording of section 1.6. In other words, if we have six floors, then the number of ways the eight people can leave is the number of nonnegative integer solutions to the problem  $x_1 + \dots + x_6 = 8$ . We know from the text that that's precisely  $\binom{8+6-1}{6-1} = \binom{13}{5} = 1287$  ways. If the reader does not have the book, look up the stars and bars method, as that's exactly what we're doing.

For the second part of the problem, we employ the same tactic across the men and the women. That is, there are  $\binom{5+6-1}{6-1} = \binom{10}{5}$  ways for the men to leave the elevator and for each of those ways there are  $\binom{3+6-1}{6-1} = \binom{8}{5}$  ways for the women to leave the elevator. There are therefore  $\binom{10}{5}\binom{8}{5} = 14112$  ways for the men and women to leave the elevator.

4. **Theoretical Exercise 8:** Prove that

$$\binom{n+m}{r} = \sum_{i=0}^r \binom{n}{i} \binom{m}{r-i}.$$

**Solution.** We use a combinatorial argument. Notice that there are  $\binom{n+m}{r}$  ways to form a committee with  $r$  people when selecting from a group with  $n$  women and  $m$  men. Then, imagine we form every possible committee with  $r - i$  men where  $0 \leq i \leq r$ . Doing so would get the terms in the sum above. We have now counted the same thing using both methods from above, so they must be equivalent.

Note: The original problem has a hint, which I omitted as I saw no need to type it out.

5. **Theoretical Exercise 11:** The following identity is known as Fermat's combinatorial identity:

$$\binom{n}{k} = \sum_{i=k}^n \binom{i-1}{k-1} \quad n \geq k.$$

Give a combinatorial argument to establish this identity. *Hint:* Consider the set of numbers 1 through  $n$ . How many subsets of size  $k$  have  $i$  as their highest-numbered member?

**Solution.** The left hand side counts the number of subsets of size  $k$  obtained from a set with  $n$  elements. For the right hand side, I imagine it will be helpful to write out a few terms:

$$\binom{k-1}{k-1} + \binom{k}{k-1} + \binom{k+1}{k-1} + \cdots + \binom{n-2}{k-1} + \binom{n-1}{k-1}.$$

It still isn't obvious how we can make progress, so let's try rewriting the above using Pascal's identity:

$$\sum_{i=k}^n \binom{i-1}{k-1} = \binom{k-1}{0} + \binom{k}{1} + \binom{k+1}{2} + \cdots + \binom{n-2}{n-k-1} + \binom{n-1}{n-k}.$$

With a little elaboration, the pattern should be clear. If we think about the given hint, we can see what each term resembles. If we wish to count the number of subsets of 1 through  $n$  in which  $k$  is the highest-numbered member, there is clearly only one way to do that: remove nothing from the previous  $k-1$  numbers. For the next term, if we want  $k+1$  to be the highest-numbered member, we must remove a number from the previous  $k$  numbers, and there are  $\binom{k}{1}$  ways to do so. Likewise, if we want  $k+2$  to be the highest-numbered member, we need to remove 2 numbers from the previous  $k+1$  numbers, and there are clearly  $\binom{k+1}{2}$  options. From here, the pattern is quite clear. If this were for a grade, we could—and probably should—provide an inductive proof that the pattern holds. However, I'm going to skip that as it's really just a formality at this point. I leave it to the reader to verify that the last terms maintain the pattern.

6. **Self Test 17:** Give an analytic verification of

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2} \quad 1 \leq k \leq n$$

**Skipped.** This is just algebra. We might need some combinatorial identities here or there, but ultimately, it's algebra. I suppose everything is sort of just algebra, but I'm not doing this problem.

7. Suppose you have 12 (distinguishable) hats and 4 (distinguishable) people. How many ways are there to divide the 12 hats among the 4 people with each person getting exactly three hats?

**Solution.** A little experimentation will help us quite a bit. Call the hats  $h_1, \dots, h_{12}$  and denote the four people by  $p_1, \dots, p_4$ . Suppose I give hats 1 through 3 to  $p_1$ , hats 4

through 6 to  $p_3$ , hats 7 through 9 to  $p_2$ , and hats 10 through 12 to  $p_1$ . Does it matter which person I choose first, second, third, and fourth to give hats to? Of course not. If I give hats 1 through 3 to  $p_4$  before anyone else has hats, that's no different than giving  $p_4$  hats 1 through 3 last. Does the order of the hats matter? Also no. I could give  $p_4$  hats 1 through 3 in six different ways, but  $p_4$  would still have hats 1 through 3 in the end. Hence, we need only think about which hats go to which person. The ordering in which we do so is irrelevant. Thus, there are

$$\binom{12}{3, 3, 3, 3} = 369,600$$

ways to divide 12 hats among 4 people, each with 3 hats.

8. Consider the permutations  $\sigma : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ .
- How many such  $\sigma$  have only one cycle? That is, how many have the property that  $\sigma(1), \sigma \circ \sigma(1), \sigma \circ \sigma \circ \sigma(1), \dots$  cycles through all  $n$  elements?
  - How many  $\sigma$  are fixed-point-free involutions? That is, how many have the property that for each  $j$ ,  $\sigma(j) \neq j$ , but  $\sigma \circ \sigma(j) = j$ .

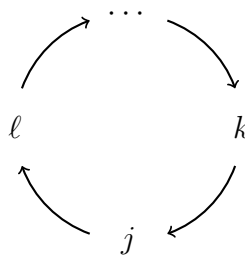
**Solution.**

- If one is familiar with cycle notation (seen in group theory), we could observe that any permutation with all  $n$  elements has the form  $(x_1 x_2 \dots x_n)$ . To find all of them, keep  $x_1$  fixed and permute the remaining  $n - 1$  elements. Clearly, there are  $(n - 1)!$  ways to do so. If one isn't familiar with group theory, the argument below also works.

Consider an arbitrary permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ x_1 & x_2 & \dots & x_n \end{pmatrix}$$

where  $\sigma(1) = x_1, \sigma(2) = x_2$  and so on. Let us first ask what happens when  $\sigma(k) = k$ . When  $\sigma(k) = k$ , we will either never reach  $k$ , or we will never reach anything else, so that isn't allowed. Now consider what happens when  $\sigma(k) = j$  and  $\sigma(j) = \ell$ , where  $\ell$  is a number we already visited. In such a case, we will clearly cycle, but we won't hit all the elements. Instead, our cycle would be



We have now concluded that in our cycle, we cannot map an element to itself, and we cannot map an element to a previously visited element. Pick any number in 1 to  $n$  and denote that as the first element in our cycle. It can be mapped to  $n - 1$  elements (any element besides itself). The next element can be mapped to  $n - 2$  elements (any element besides itself and the first element), and so on until we get to the  $n - 2$  element. The  $n - 2$  element has only one option,  $n - 1$ , and  $n - 1$  maps to the first element. We now see that there are  $(n - 1)!$  permutations that have only one cycle.

- b) In order for  $\sigma(j) \neq j$  and  $\sigma \circ \sigma(j) = j$  to hold, each  $j$  must have a corresponding  $k$  such that  $\sigma(j) = k$  and  $\sigma(k) = j$ . Thus,  $n$  must be even. There are  $\binom{n}{2}$  options for the first pair,  $\binom{n-2}{2}$  for the second pair, and so on, which is equivalent to

$$\binom{n}{2, 2, \dots, 2}.$$