The N-Dulum

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1 Abstract

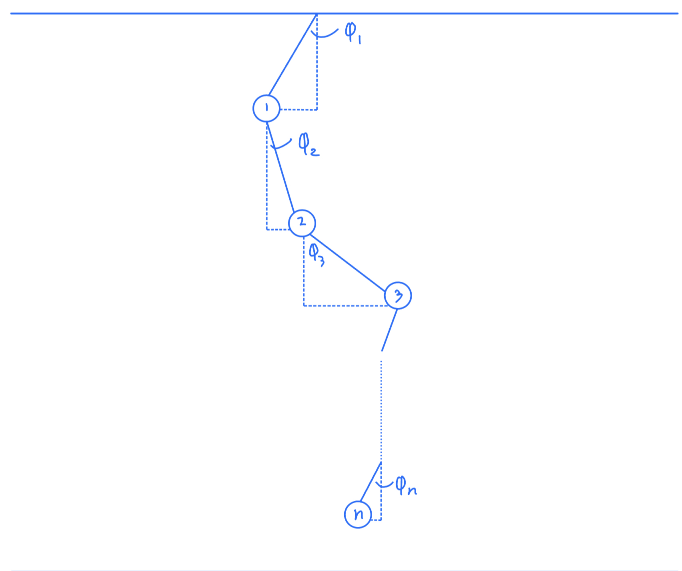
In this paper we will explore the motion of a pendulum with n masses by composing a Lagrangian and using the Lagrange Equations to find equations of motion for each mass. We will treat the masses as point-like objects and we will assume that the length of the pendulum and its point of support do not change (and consequently, the distance between masses is the same for every mass). Our setup will enable us to chain together the positions of the masses, which will allow for an iterative representation of the equations of motion. Meaning, for each mass m_i in the N-dulum, our Lagrangian will be capable of producing a corresponding equation of motion that, when solved, would produce a postion function $\phi_i(t)$.

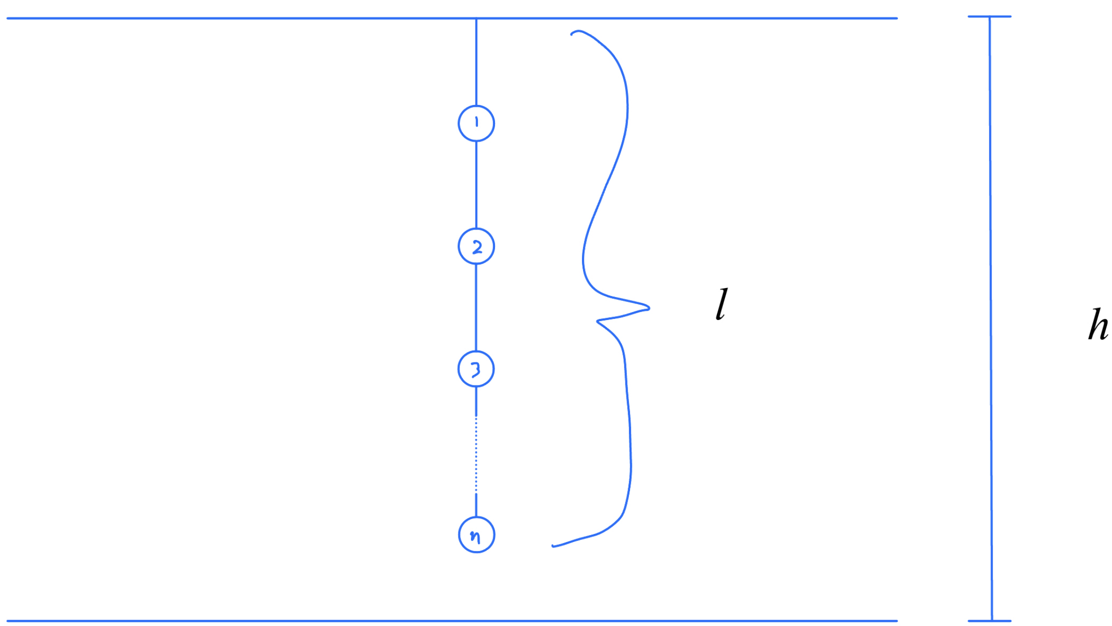
2 Introduction

To begin, we need to form a conceptual understanding of our system. We will denote the length of the pendulum as l and we will let l be a constant. Since l is constant, the distance between masses (on the N-dulum) is always l/n where n is the number of masses. Without this constraint, the problem could be considerably more difficult. As we shall see later, one of the core ideas to our approach is to make use of the power of symmetry. Another important constant is the height between the top of the pendulum (point of support) and the ground; we denote this height as h. There will be some that point out that we don't actually need the constant h. However, when we compute gravitational potential energy, I find it much easier to visualize what we are doing and why by introducing a constant (more on that later). Though we don't need to, our computation will be less error prone if we let all masses on the N-dulum have the same magnitude. We will measure the position of mass m_i wtih ϕ_i, the angle between a "sub-dulum" s_i and the vertical. If it is not clear, the sub-dulum s_i is simply the i^{th} pendulum. That is, on a double pendulum, we have the first pendulum starting at the point of support and ending at the first mass whereas the second pendulum starts at the first mass and ends at the second mass. Likewise, on the N-dulum, the i^{th} pendulum s_i starts at the mass m_{i-1} and ends at m_i.

We are almost ready to begin, but there are a two key considerations worth mentioning. First, we assume the entire N-dulum is free to move without friction in a plane. Ignoring friction allows us to be confident we are dealing with a similar system to the single and double pendulum from chapter 7 and thus removes any need to worry about whether we can use Lagrangian methods. Second, we assume that the rod of each s_i is massless. Such an assumption will allow for a nice interpretation of arbitrarily many masses. Namely, as we let $n \to \infty$, the N-dulum begins to behave like a rope with mass nm.

We are now ready to start considering the N-dulum itself. Given some pendulum with a length l, a point of support with a height h above the ground, and n masses, we have a system like so:





3 Composing Lagrangian

Our first step in forming the Lagrangian is recognizing that the kinetic and potential energy of the N-dulum can each be written as a sum of the respective energies of each sub-dulum. That is

$$T = \sum_{i=1}^{n} T_{s_i} \text{ and } U = \sum_{i=1}^{n} U_{s_i}.$$

3.1 Kinetic Energy

We know kinetic energy is given by $\frac{1}{2}mv^2$ so we should expect that if m_i has a position of $\mathbf{r}_i$ then the kinetic energy of each sub-dulum s_i is given by $\frac{1}{2}m\dot{\mathbf{r}}_i^2$. Thus, the total kinetic energy T is written as

$$T = \sum_{i=1}^{n} \frac{1}{2}m\dot{\mathbf{r}}_i^2.$$

It follows that we will eventually need to understand the term $\sum \dot{\mathbf{r}}_i^2$. We will first need to understand $\mathbf{r}_i$, however.

3.1.1 Position Vectors

Note about unit vectors: Later I will point out the importance of not underestimating trigonometry. That turned out to be painfully ironic. When making my calculations, I dropped vector notation so as to save space (seeing everything in a neat line helps me think). My work is correct, but if and only if we work in a plane that doesn't follow what we are used to. That is, if we just plug in the basis vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ our heights will be incorrect. I measured everything as if our origin is at the point of support. Meaning, the positive vertical direction is going down, and the positive horizontal direction is going to the right. I originally planned to use $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ to denote vector functions, but I clearly cannot. Thus, we will define a new unit vector $\hat{\mathbf{J}}$ which points in the negative vertical direction on the cartesian plane (positive vertical direction for our system).

From the diagram above, we can see that the position vector for m_1 is

$$\mathbf{r}_1(t) = \frac{l}{n}\sin\phi_1\hat{\mathbf{i}} + \frac{l}{n}\cos\phi_1\hat{\mathbf{J}} \tag{1}$$

where ϕ_1 is a function of time [1]. Though it may seem trivial, it's very important to understand how we derive the position vector $\mathbf{r}_1(t)$ as the ideas used to do so will be chained together to form every subsequent position vector (power of symmetry). Thus, if we can understand a single postion vector, we can understand all postion vectors; we have now justified the motivation for explicitly stating the meaning of the terms in equation (1).

Recall from trigonometry that for a right triangle, we can find the values of the base and height of the triangle if we have an angle and the hypotenuse. In this case, we are considering the triangle where $\frac{l}{n}$ is the hypotenuse, the angle we are considering is ϕ_1, and the base and height are the horizontal and vertical components of $\mathbf{r}_1(t)$ respectively. We obtain the value of the horizontal component with the equation $\frac{x_{comp}}{l/n} = \sin\phi_1$. Likewise, we obtain the value of the vertical component with $\frac{y_{comp}}{l/n} = \cos\phi_1$. Then, we simply scale the $\hat{\mathbf{i}}$ and $\hat{\mathbf{J}}$ vectors by $\frac{l}{n}\sin\phi_1$ and $\frac{l}{n}\cos\phi_1$ respectively. The importance of all that computation lies in our consideration of $\mathbf{r}_i(t)$ for $i > 1$. For instance, we find $\mathbf{r}_2(t)$ with

$$\mathbf{r}_2(t) = \mathbf{r}_1(t) + \frac{l}{n}\big(\sin\phi_2\hat{\mathbf{i}} + \cos\phi_2\hat{\mathbf{J}}\big)$$

which can be rewritten as

$$\mathbf{r}_2(t) = \frac{l}{n}\sin\phi_1\hat{\mathbf{i}} + \frac{l}{n}\sin\phi_2\hat{\mathbf{i}} + \frac{l}{n}\cos\phi_1\hat{\mathbf{J}} + \frac{l}{n}\cos\phi_2\hat{\mathbf{J}} \tag{2}$$

where ϕ_2, and more generally, ϕ_i, is also a function of time. Equation (2) should demonstrate our seemingly unnecessary review of SOH CAH TOA. When finding $\mathbf{r}_2(t)$, we are adding up the total vertical and horizontal distance necessary to get to m_2. That is, after we have found $\mathbf{r}_1(t)$, we set m_1 as our new, temporary origin and find $\mathbf{r}_2(t)$ exactly like we found $\mathbf{r}_1(t)$. We can then do the same thing for $\mathbf{r}_3(t)$ (treat m_2 as new origin) and so on for any $\mathbf{r}_i(t)$ [2]. We can now see that the position vector for the i^{th} mass is

$$\mathbf{r}_i(t) = \frac{l}{n}\Bigg(\big(\sin\phi_1 + \sin\phi_2 + \cdots + \sin\phi_i\big)\hat{\mathbf{i}} + \big(\cos\phi_1 + \cos\phi_2 + \cdots + \cos\phi_i\big)\hat{\mathbf{J}}\Bigg) \tag{3}$$

3.1.2 Squaring the Velocity Vectors

Now that we have position vectors, we can differentiate to get velocity vectors:

$$\begin{aligned}
\dot{\mathbf{r}}_1 &= \frac{l}{n}\Bigg(\Big(\cos(\phi_1)\dot{\phi_1}\Big)\hat{\mathbf{i}} + \Big(-\sin(\phi_1)\dot{\phi_1}\Big)\hat{\mathbf{J}}\Bigg)\\
\dot{\mathbf{r}}_2 &= \frac{l}{n}\Bigg(\Big(\cos(\phi_1)\dot{\phi_1} + \cos(\phi_2)\dot{\phi_2}\Big)\hat{\mathbf{i}} + \Big(-\sin(\phi_1)\dot{\phi_1} - \sin(\phi_2)\dot{\phi_2}\Big)\hat{\mathbf{J}}\Bigg)\\
&\vdots\\
\dot{\mathbf{r}}_i &= \frac{l}{n}\Bigg[\Big(\sum_{i'=1}^{i}\cos(\phi_{i'})\dot{\phi_{i'}}\Big)\hat{\mathbf{i}} + \Big(\sum_{i'=1}^{i} -\sin(\phi_{i'})\dot{\phi_{i'}}\Big)\hat{\mathbf{J}}\Bigg]
\end{aligned}$$

One of our last steps in understanding $\sum \dot{\mathbf{r}}_i^2$ is to simply compute the first few $\dot{\mathbf{r}}_i^2$ terms [3]. Notice that we don't yet have a formula for $\dot{\mathbf{r}}_i^2$ so it would be rather difficult to compute $\sum \dot{\mathbf{r}}_i^2$. While computing $\dot{\mathbf{r}}_i^2$, it will be helpful to remember $\sin^2\theta + \cos^2\theta = 1$ and $\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2$. Let us begin the computation of $\dot{\mathbf{r}}_i^2$. We first compute $\dot{\mathbf{r}}_1^2$:

$$\begin{aligned}\dot{\mathbf{r}}_1^2 &= \frac{l^2}{n^2}\big(\cos^2(\phi_1)\dot{\phi_1}^2 + \sin^2(\phi_1)\dot{\phi_1}^2\big)\\ &= \frac{l^2}{n^2}\dot{\phi}_1^2\end{aligned}$$

Next, $\dot{\mathbf{r}}_2^2$:

$$\begin{aligned}\dot{\mathbf{r}}_2^2 = \frac{l^2}{n^2}\Big[&\cos^2(\phi_1)\dot{\phi_1}^2 + \cos^2(\phi_2)\dot{\phi_2}^2 + 2\dot{\phi}_1\dot{\phi}_2\cos\phi_1\cos\phi_2\\ &+\sin^2(\phi_1)\dot{\phi_1}^2 + \sin^2(\phi_2)\dot{\phi_2}^2 + 2\dot{\phi}_1\dot{\phi}_2\sin\phi_1\sin\phi_2\Big]\\ = \frac{l^2}{n^2}&\Big(\dot{\phi_1}^2 + \dot{\phi_2}^2 + 2\big[\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)\big]\Big)\end{aligned}$$

And after $\dot{\mathbf{r}}_3^2$, the pattern should be clear:

$$\begin{aligned}\dot{\mathbf{r}}_3^2 = \frac{l^2}{n^2}\Big[&\cos^2(\phi_1)\dot{\phi_1}^2 + \cos^2(\phi_2)\dot{\phi_2}^2 + \cos^2(\phi_3)\dot{\phi_3}^2\\ &+2\dot{\phi}_1\dot{\phi}_2\cos\phi_1\cos\phi_2 + 2\dot{\phi}_1\dot{\phi}_3\cos\phi_1\cos\phi_3 + 2\dot{\phi}_2\dot{\phi}_3\cos\phi_2\cos\phi_3\\ &+\sin^2(\phi_1)\dot{\phi_1}^2 + \sin^2(\phi_2)\dot{\phi_2}^2 + \sin^2(\phi_3)\dot{\phi_3}^2\\ &+2\dot{\phi}_1\dot{\phi}_2\sin\phi_1\sin\phi_2 + 2\dot{\phi}_1\dot{\phi}_3\sin\phi_1\sin\phi_3 + 2\dot{\phi}_2\dot{\phi}_3\sin\phi_2\sin\phi_3\Big]\\ = \frac{l^2}{n^2}&\Big(\dot{\phi_1}^2 + \dot{\phi_2}^2 + \dot{\phi_3}^2 + 2\big[\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2) + \dot{\phi}_1\dot{\phi}_3\cos(\phi_1-\phi_3) + \dot{\phi}_2\dot{\phi}_3\cos(\phi_2-\phi_3)\big]\Big)\end{aligned}$$

We can now see that for $\dot{\mathbf{r}}_i^2$ (where $i > 1$) we have

$$\dot{\mathbf{r}}_i^2 = \frac{l^2}{n^2}\Big[\sum_{j=1}^{i}\dot{\phi_j}^2 + 2\sum_{j=1}^{i-1}\sum_{k=j+1}^{i}\dot{\phi}_j\dot{\phi}_k\cos(\phi_j-\phi_k)\Big].$$

We can now sum up the velocity squared terms to get kinetic energy. Our lengthy computation yields

$$
\begin{aligned}
T &= \frac{1}{2}m\sum_{i=1}^{n}\dot{\mathbf{r}}_i^2 \\
&= \frac{1}{2}m\sum_{i=2}^{n}\left[\frac{l^2}{n^2}\Big[\sum_{j=1}^{i}\dot{\phi_j}^2 + 2\sum_{j=1}^{i-1}\sum_{k=j+1}^{i}\dot{\phi}_j\dot{\phi}_k\cos(\phi_j-\phi_k)\Big]\right] + \frac{ml^2}{2n^2}\dot{\phi_1}^2 \\
&= \frac{1}{2}m\frac{l^2}{n^2}\sum_{i=2}^{n}\left[\Big[\sum_{j=1}^{i}\dot{\phi_j}^2 + 2\sum_{j=1}^{i-1}\sum_{k=j+1}^{i}\dot{\phi}_j\dot{\phi}_k\cos(\phi_j-\phi_k)\Big]\right] + +\frac{ml^2}{2n^2}\dot{\phi_1}^2 \\
&= \frac{ml^2}{2n^2}\Bigg[\left(\dot{\phi}_1^2\right) + \left(\dot{\phi_1}^2 + \dot{\phi_2}^2 + 2\big[\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)\big]\right) \\
&+ \left(\dot{\phi_1}^2 + \dot{\phi_2}^2 + \dot{\phi_3}^2 + 2\big[\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2) + \dot{\phi}_1\dot{\phi}_3\cos(\phi_1-\phi_3) + \dot{\phi}_2\dot{\phi}_3\cos(\phi_2-\phi_3)\big]\right) \\
&+ \Bigg(\sum_{j=1}^{4}\dot{\phi_j}^2 + 2\big[\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2) + \dot{\phi}_1\dot{\phi}_3\cos(\phi_1-\phi_3) + \dot{\phi}_1\dot{\phi}_4\cos(\phi_1-\phi_4) \\
&+ \dot{\phi}_2\dot{\phi}_3\cos(\phi_2-\phi_3) + \dot{\phi}_2\dot{\phi}_4\cos(\phi_2-\phi_4) + \dot{\phi}_3\dot{\phi}_4\cos(\phi_3-\phi_4)\Bigg) + \cdots \\
&\cdots + \left(\sum_{j=1}^{n}\dot{\phi_j}^2 + 2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\dot{\phi}_j\dot{\phi}_k\cos(\phi_j-\phi_k)\big]\right)\Bigg]
\end{aligned}
$$

In the last "line" (everything after the last equals sign) we have what appears to be a mess but isn't actually that bad. One might notice the comically large parentheses. They are large on purpose. Each parenthesis...group [4] is an iteration of $\sum_{i=1}^{n}$. Therefore, the kinetic energy is given by

$$
T = \frac{ml^2}{2n^2}\left[\sum_{j=1}^{n}\zeta_j\dot{\phi}_j^2 + 2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\zeta_k\dot{\phi}_j\dot{\phi}_k\cos(\phi_j-\phi_k)\right]
$$

where $\zeta_j = n+1-j$. The ζ_j term comes from the chaining of terms. Meaning, the position of the i^{th} mass depends on the position of the $i^{th}-1$ mass. As such, every iteration will have a $\dot{\phi}_1^2$ term, every iteration after the first will have a $\dot{\phi}_2^2$ term, and so on. The $\dot{\phi}_j\dot{\phi}_k\cos(\phi_j-\phi_k)$ terms are slightly different. The $\dot{\phi}_j\dot{\phi}_k\cos(\phi_j-\phi_k)$ are biproducts of squaring a sum of terms. That is, $(a+b)^2 = a^2+b^2+2ab$ and $(a+b+c)^2 = a^2+b^2+c^2+ab+ac+bc$. We can see that the $\dot{\phi}_j\dot{\phi}_k\cos(\phi_j-\phi_k)$ are analogous to the ab, ac, bc terms so in order for the $\dot{\phi}_j\dot{\phi}_k\cos(\phi_j-\phi_k)$ terms to appear, we need $i>1$. Other than that, the same reasoning applied to the derivation of ζ_j can be applied to ζ_k. When we count the terms up, we see that $\zeta_k = n+1-k$.

3.2 Potential Energy

Recall that the position of the i^{th} mass is

$$\mathbf{r}_i(t) = \frac{l}{n}\Bigg(\big(\sin\phi_1 + \sin\phi_2 + \cdots + \sin\phi_i\big)\hat{\mathbf{i}} + \big(\cos\phi_1 + \cos\phi_2 + \cdots + \cos\phi_i\big)\hat{\mathbf{J}}\Bigg).$$

Since the height between the top of the pendulum and the ground is some value h we can write the GPE of the i^{th} mass as $mg(h - y_i)$ where $y_i = \sum_{j=1}^{i} \cos\phi_j$. The total potential energy is simply the sum of all $mg(h - y_i)$. By analyzing the sum of the first few terms, we should be confident that

$$U = mg\Bigg[nh - \sum_{j=1}^{n} \frac{\zeta_j l}{n}\cos\phi_j\Bigg].$$

Observe that the sum of the first three terms is

$$\begin{aligned}
U_3 &= mg(h - y_1) + mg(h - y_2) + mg(h - y_3)\\
&= mg(3h - y_1 - y_2 - y_3)\\
&= mg\left[3h - \frac{l}{n}\cos\phi_1 - \left(\frac{l}{n}\cos\phi_1 + \frac{l}{n}\cos\phi_2\right) - \left(\frac{l}{n}\cos\phi_1 + \frac{l}{n}\cos\phi_2 + \frac{l}{n}\cos\phi_3\right)\right]\\
&= mg\left[3h - \left(\frac{3l}{n}\cos\phi_1 + \frac{2l}{n}\cos\phi_2 + \frac{l}{n}\cos\phi_3\right)\right].
\end{aligned}$$

From here, we get U_4 with

$$\begin{aligned}
U_4 &= U_3 + y_4\\
&= U_3 + mg\left[h - \left(\frac{l}{n}\cos\phi_1 + \frac{l}{n}\cos\phi_2 + \frac{l}{n}\cos\phi_3 + \frac{l}{n}\cos\phi_4\right)\right]\\
&= mg\left[4h - \left(\frac{4l}{n}\cos\phi_1 + \frac{3l}{n}\cos\phi_2 + \frac{2l}{n}\cos\phi_3 + \frac{l}{n}\cos\phi_4\right)\right]\\
&= mg\Bigg[4h - \sum_{j=1}^{4} \frac{(4 + 1 - j)l}{n}\cos\phi_j\Bigg]
\end{aligned}$$

and more generally

$$
\begin{aligned}
U_i &= U_{i-1} + mg\Big[h - \sum_{j=1}^{i} \frac{l}{n}\cos\phi_j\Big] \\
&= mg\Big[ih - \sum_{j}^{i} \frac{(i+1-j)l}{n}\cos\phi_j\Big]
\end{aligned}
$$

which implies

$$
\begin{aligned}
U &= U_n \\
&= mg\Big[nh - \sum_{j}^{n} \frac{l(n+1-j)}{n}\cos\phi_j\Big] \\
&= mg\Big[nh - \sum_{j}^{n} \frac{\zeta_j l}{n}\cos\phi_j\Big]
\end{aligned}
$$

as desired.

4 Lagrangian and Lagrange Equations

Putting all of our previous work together, we have

$$
\mathcal{L} = \frac{1}{2}m\frac{l^2}{n^2}\left[\sum_{j=1}^{n}\zeta_j\dot{\phi_j}^2 + 2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\zeta_k\dot{\phi}_j\dot{\phi}_k\cos(\phi_j-\phi_k)\right] - mg\left[nh - \sum_{j=1}^{n}\frac{\zeta_j l}{n}\cos\phi_j\right].
$$

We have a fair bit going on in our Lagrangian so we want to be careful and make sure that we don't miss any terms. In short, we are computing n LE equations. That is, we are looking for

$$
\frac{\partial\mathcal{L}}{\partial\phi_i} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_i}
$$

for all i such that $i \leq n$ and $i \in \mathbb{N}$. Unfortunately, there isn't really a clean way to do this. We will just have to take it one step at a time and account for all cases.

4.1 Partial with respect to Phi

Case 1:

There's a lot we need to consider in all three cases, so we will write a copy of the Lagrangian:

$$\mathcal{L} = \frac{1}{2}m\frac{l^2}{n^2}\left[\sum_{j=1}^{n}\zeta_j\dot{\phi_j}^2 + 2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\zeta_k\dot{\phi_j}\dot{\phi_k}\cos(\phi_j-\phi_k)\right] - mg\left[nh - \sum_{j=1}^{n}\frac{\zeta_j l}{n}\cos\phi_j\right].$$

In the first case, we are looking for $\frac{\partial\mathcal{L}}{\partial\phi_1}$. Notice that in the offset double sum, only j can be equal to 1. Therefore, while computing $\frac{\partial\mathcal{L}}{\partial\phi_1}$ our double sum becomes a single sum. That is, we hold j at 1 (so that we don't miss any $j = 1$ terms) and compute the partial derivative. Such an action gives

$$\frac{\partial\mathcal{L}}{\partial\phi_1} = -\frac{ml^2}{n^2}\left[\sum_{k=2}^{n}(n+1-k)\dot{\phi_1}\dot{\phi_k}\sin(\phi_1-\phi_k)\right] - mgl\sin\phi_1.$$

You'll notice I removed ζ_k and replaced it with its actual meaning as to avoid confusion. It's critical we don't mix up our subscripts here so it's good to write out as much as possible.

Case 2: In case 2 we want to consider what happens for $2 \le i \le n-1$. Notice that both j and k exist in the range 2 to $n-1$. Therefore, we need to comput two partial derivatives. We need one partial for the occurences where $j = i$ and one partial for $k = i$. The sum of both partials will give $\frac{\partial\mathcal{L}}{\partial\phi_i}$. We are going to be just as annoying as Taylor and skip heaps of algebra to get

$$\frac{\partial\mathcal{L}}{\partial\phi_i} = -\frac{ml^2}{n^2}\left[\sum_{k=i}^{n}(n+1-k)\dot{\phi_i}\dot{\phi_k}\sin(\phi_i-\phi_k) - \sum_{j=1}^{i-1}(n+1-i)\dot{\phi_j}\dot{\phi_i}\sin(\phi_j-\phi_i)\right] - mg\frac{(n+1-i)l}{n}\sin\phi_i.$$

Again, I removed the ζ notation and we see that when j is fixed, the j subscripts are i and when k is fixed, the k subscripts are i (for $2 \le i \le n-1$).

Case 3:

In case 3, we are looking for $\frac{\partial\mathcal{L}}{\partial\phi_n}$ which, in Kinetic energy, only occurs in k terms. Therefore, we will hold the k sum constant at $k = n$ and differentiate to get

$$\frac{\partial\mathcal{L}}{\partial\phi_n} = \frac{ml^2}{n^2} = \left[\sum_{j}^{n-1}\dot{\phi_j}\dot{\phi_n}\sin(\phi_j-\phi_n) - mg\sin(\phi_n)\right].$$

Notice the lack of a negative sign in front of the $\frac{ml^2}{n^2}$ term; that is not a mistake. This time around, the negatives from the derivative of cos and the ϕ_i term cancel out.

4.2 Partial with respect to Phi dot

When computing the partial derivatives with respect to the time derivative of $\dot{\phi}_i$, we will have the same offset problem. We will use the same methodology. That is, we will consider $i = 1$, $2 \leq i \leq n-1$, and $i = n$. After all computation is said and done, we have

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} = \frac{ml^2}{n^2}\left[n\dot{\phi}_1 + \left(\sum_{k=2}^{n}(n+1-k)\dot{\phi}_k \cos(\phi_1 - \phi_k)\right)\right]$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} = \frac{ml^2}{n^2}\left[(n+1-i)\dot{\phi}_i + \left(\sum_{k=i+1}^{n}\dot{\phi}_k \cos(\phi_i - \phi_k) + \sum_{j=1}^{i-1}(n+1-i)\dot{\phi}_j \cos(\phi_j - \phi_i)\right)\right]$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}_n} = \frac{ml^2}{n^2}\left[\dot{\phi}_n + \left(\sum_{j=1}^{n-1}\dot{\phi}_j \cos(\phi_j - \phi_n)\right)\right].$$

4.3 Time derivative of Partials

We now need to compute the time derivatives of the partials we just found. The differentiation yields

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} = \frac{ml^2}{n^2}\left[n\ddot{\phi}_1 + \left(\sum_{k=2}^{n}\Big((n+1-k)\ddot{\phi}_k \cos(\phi_1 - \phi_k) - \dot{\phi}_k \sin(\phi_1 - \phi_k)(\dot{\phi}_1 - \dot{\phi}_k)\Big)\right)\right]$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} = \frac{ml^2}{n^2}\Bigg[(n+1-i)\ddot{\phi}_i + \left(\sum_{k=i+1}^{n}\Big(\ddot{\phi}_k \cos(\phi_i - \phi_k) - \dot{\phi}_k \sin(\phi_i - \phi_k)(\dot{\phi}_i - \dot{\phi}_k)\Big)\right)$$
$$+\left(\sum_{j=1}^{i-1}\Big((n+1-i)\ddot{\phi}_j \cos(\phi_j - \phi_i) - \dot{\phi}_j \sin(\phi_j - \phi_i)(\dot{\phi}_j - \dot{\phi}_i)\Big)\right)\Bigg]$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_n} = \frac{ml^2}{n^2}\left[\ddot{\phi}_n + \left(\sum_{j=1}^{n-1}\left(\ddot{\phi}_j\cos(\phi_j-\phi_n) - \dot{\phi}_j\sin(\phi_j-\phi_n)(\dot{\phi}_j-\dot{\phi}_n)\right)\right)\right].$$

We have now found everything needed to compose the Lagrange equations. However, writing them out would not be particularly illuminating as they are what one might call "messy".

5 Conclusion

Our analysis of the N-dulum serves a few goals. Above *almost* all else, we wish to have fun. Truthfully, the N-dulum is not what one might call "practical". While it is true that for higher numbers of n the N-dulum behaves closer and closer to a rope/chain, I have no idea what kind of computing power would be needed to simulate such a system. Considering computational power is something I would have done had I had time to do any coding, but unfortunately, that has not (yet) happened. In addition to intellectual play, we wish to practice problem solving. The N-dulum—and any system that involves generalization, for that matter—exists as a wonderful opportunity to test our ability to conceptually understand a problem and the tools we need to solve it. I find that in a problem that requires exceptionally generalized thinking, one needs to be more careful about the finer details. For instance, should we be worried about angles to the left of the vertical? If we were drawing by hand, we could just denote such an angle as $-\phi$, but in the general case, it would be difficult to implement such a strategy. If my understanding is correct, we do not need to worry about angles to the left of the vertical. That is, our computation still provides the correct answer because we can consider some angle $-\phi$ to be equivalent to $2\pi - \phi$ and we should get the same answer with both angles [5]. Litte sub-problems like this arise constantly when dealing with a generic model and I believe that that's what makes generalization valueable. Generalization forces one to question what they think they already know. In working through this project, I've had to revisit trigonemtirc properties, algebraic properties, dot products, the least action principle, conservative forces, constraints, the conceptual fundamentals of GPE, derivatives, numerical solutions to differential equations, and induction—to name a few topics.

In case it is not already abundantly clear, this project is far from finished. I have not coded up solutions. I do not have an animated visual. My algebra, where it exists, could likely be explained in a clearer way. Among the many topics I would like to address when my work here is finally done are normal modes and Newtonian solutions to the N-dulum (where I would consider incorporating friction, if I were really feeling crazy). Also, though I have indirectly said so already, I really do want a solution to the N-dulum. If possible I would love to adjust a slider and see results in real time (solution and animation). I look forward to harassing Jeremy with my future questions.

6 Acknowledgements

Speaking of Jeremy, he's my only source for this project. I did skim another paper, but that skim occured after my results were achieved and it lasted for about 30 seconds. I would need a more extensive reading (with pen and paper) to really understand the differences between my work and Ryan's. As such, a bibliography didn't seem appropriate. Rather, I think I owe Jeremy a thank you. We agreed to meet once a week and towards the end of the semester it seemed I was in his office everyday.

thanks my guy

Notes

[1]Is there a difference between saying "of time" and "with respect to time"?

[2]At some point I will learn Tikz because I really want to show what's happening with vector addition. I think it's important that we understand why the chaining of the position vectors works, because without that relationship, I imagine the problem would be considerably less intuitive. Edit: I have learned Tikz (a little). Unfortunately, I am not proficient enough to be fast, so we have very ugly hand made drawings in my deliverable.

[3]I want to include something about dot product (avoiding the freshman's dream is kinda important). Also, check this out $\dot{\phi_1}^2$ vs. $\dot{\phi}_1^2$

[4]I don't think "group" is the right word. I'm hesitant to use "term" because I think that could be confusing. Anyway.

[5]I still wonder if this could pose for some problems when writing code to generate a visual, but whatever. Problem for another day.

Edit: I left my notes in because I don't really consider my deliverable deliverable. It's more of a draft. Being that that's the case, I figured I might as well. It seems to me that I'll learn more if I highlight some of my ignorances/mistakes/random thoughts. Some of the notes might be gibberish (too tired to check).